Mathematics in Art: Color Symmetries of Pampanga's Giant Christmas Lanterns

Imogene F. Evidente, Ph.D. and Angela D. Carreon, M.A.
University of the Philippines Diliman
Extension Program in Pampanga
Dedicated to Professor René P. Felix

ABSTRACT

One of the marks of Kapampangan art and culture is the Giant Christmas Lantern. Every Christmas season, giant lanterns designed by Pampanga's artists are displayed during the Giant Lantern Festival, known locally as "Ligligan Parul". In the Kapampangan vernacular, "parul" and "ligligan" mean lantern and competition respectively. In essence, the term "Ligligan Parul," embodies the contest among different barangays of creating the best giant lantern in the province. The colorful and symmetric designs of these giant lanterns have an entrancing effect and have become more and more intricate through the years.

This contribution explores the giant lantern as *math art*. The underlying uncolored designs of giant lanterns are simple symmetrical patterns whose mathematical structures are well known. There are various mathematical methods for coming up with the symmetric colorings of such patterns. We investigate the color symmetry of existing giant parol designs by matching them with color symmetry algorithms that make use of the subgroup structure of the symmetry group of a pattern. Finally, we demonstrate how to obtain symmetric colorings of the giant lantern that do not match any of the existing designs.

Keywords: Pampanga giant lantern, parol, symmetry, color symmetry, coset coloring

(1) INTRODUCTION AND OUTLINE

"There is geometry in the humming of the strings. There is music in the spacing of the spheres," Pythagoras once said. For this ancient philosopher and mathematician, the elegance of mathematics was evident in the beauty and harmony in art. However, this link between math and art remains obscure to many, even though math has always appeared in various art forms. For instance, math appears in classical art's emphasis on symmetry and proportion, in the way the Asian art of origami adheres to the rules of geometry, and in the use of algorithms to generate 21st century digital art.

The relationship between math and art is reciprocal: art serves math, and math serves art. Scientists use art to explore and express their ideas. Art can demonstrate the elegance of mathematical theory and provides mathematicians a way to make the subject more accessible. Students are more receptive when they are able to visualize abstract concepts, especially if this is achieved through beautiful artworks.

Conversely, math is a servant of the arts. A formal analysis of art through math reveals shifts in schools of thought (Makovicky, 1986). Practice-based artists also use math to develop their techniques (Happerset, 2020). One of the most renowned of such artists is M.C. Escher, a 20th-century Dutch artist whose fascination with the Moorish patterns in the Alhambra led him to dive deep into complex math concepts. This allowed him to eventually create some of the most mind-bending images of 20th century art (Taschen, 2016). Susan Happersett, an artist whose love for math and art has led to her mission of changing the minds of those adverse to math through her art, creates rule-based art using mathematical algorithms (Happersett, 2020). In fact, we are seeing a new generation of artists who grew up with technology and are utilizing algorithmic processing to create digital art. As we also move towards a society where interdisciplinary studies are emphasized, math in art enables educators from different art fields to inject STEAM into their lessons (Happersett, 2020).

Collaboration between artists and mathematicians over the years reflects the reciprocal relationship between math and art. Many universities organize conferences where mathematicians and artists come together to explore the math and art connection. The most popular among these is the Bridges Conference, where mathematicians and artists from around the world meet every year to exchange ideas on the latest trends in math art. *The Journal of Mathematics and the Arts* is another result of the collaboration between artists and mathematicians. This peer-reviewed journal was established in 2007 to provide mathematicians and artists a venue to publish work related to math art. It is designed to be "a place for those engaged in using mathematics in the creation of works of art, who seem to understand art arising from math or scientific endeavors, who strive to explore the mathematical implications of artistic works" (Journal of Mathematics and the Arts, n.d.).

Unfortunately, math art has hardly been explored in the Philippines. One possible reason is that many, especially those engaged in the arts, still view math as a dreaded subject to be avoided at all costs. Furthermore, only a handful of Filipino mathematicians are doing formal research in this area. Most of these mathematicians can trace their academic genealogy back to Professor René P. Felix, who pioneered the study of symmetry and color symmetry in the country. One of his students led a group that analyzed crystallographic patterns in Philippine Indigenous textiles (De Las Peñas et al., 2018). But apart from a few such studies, it is rare for artists and mathematicians to collaborate in the Philippines.

Through this contribution, we wish to promote the study of math art in the Philippines. In particular, our goals are: (1) to promote the study of math as it appears in Philippine art, (2) to introduce the formal study of color symmetry to the mathematicians of Central Luzon, (3) to encourage the use of art in instruction, especially in the teaching of abstract algebra to math majors, (4) to present to artists, especially those engaged in graphic and digital art, the possibility of using math to generate art, and (5) to motivate mathematicians and artists to collaborate in studying and producing math art.

To achieve these goals, we demonstrate color symmetry methods by applying them to Pampanga's giant lanterns. We hope that by choosing to apply color symmetry to a source of pride deeply rooted in the cultural heritage of Central Luzon, we are able to capture the interest of our target audience.

The designs of the giant lanterns are always symmetric. This is not surprising since symmetry, being pleasing to the eye, appears in many art forms (Conway et al., 2008). Even though the designs of the giant lanterns have become increasingly elaborate over the years, the designs largely remain centered on the five-pointed star. Thus, the giant lanterns are simple symmetrical patterns whose mathematical structures are well-known. Furthermore, there are established mathematical methods for coloring such structures.

This article is organized as follows. Section 2 describes the methodology used to arrive at the results. We give a brief history of the art of parol making in Section 3. Section 4 discusses the design of the parol from the point of view of the artists and craftsmen. Meanwhile, Section 5 provides a theoretical discussion on the structure of the Pampanga giant lantern from the perspective of mathematical symmetry. In Section 6, we present a coloring method using the left and right cosets of a subgroup of the symmetry group of a pattern. We illustrate the algorithm by applying it to some designs of the Pampanga giant lantern. We end in Section 7 by summarizing the analysis and suggesting some areas for further research.

We have written this article in a way that makes it accessible to the various segments of our target audience. We use the language of mathematicians since we wish to demonstrate how art can be used in formal math instruction. However, we define concepts in a way that even those with only a basic knowledge of math may follow the discussion. We also give illustrations of the various math concepts and methods using a simple pointed star before applying them to the giant lantern. Given the goals of this contribution, we do not focus on justifying and proving the theory supporting these colorings. Those interested may consult the references for a deeper insight into the theory of color symmetry, specifically left coset colorings. A theoretical discussion of right coset colorings will be published elsewhere.

(2) METHODOLOGY

We analyzed the color symmetry of the giant lantern using standard techniques in the study of symmetry. We began by determining the mathematical structure of the uncolored giant lantern. We inspected recent designs of giant lanterns exhibited in various Giant Lantern Festivals. We also examined the designs in the Parul Sampernandu Coloring Book (City of San Fernando Pampanga Tourism Office, 2015), a collection of uncolored giant lantern designs by some of Pampanga's most acclaimed parol artists. Once the mathematical structure of the uncolored pattern was established, we proceeded to analyze the colored designs. We matched the colored patterns with some color symmetry algorithm that would give rise to the colored pattern. Finally, we attempted to come up with symmetric colorings of the giant lantern not similar to existing designs. We did this by altering the parameters of the color symmetry algorithm that matched existing designs, or by applying a color symmetry algorithm that did not match the existing designs. To supplement our analysis, we also interviewed some of the giant parol artists to gain insight into how they come up with their designs.

(3) HISTORY OF THE ART OF PAROL MAKING

The lantern making tradition in San Fernando, Pampanga traces back its origins to Spanish colonists in Bacolor, the old capital of Pampanga, who urged people to hold lantern processions honoring the Virgin Mary, "Our Lady of La Naval" (dela Cruz, 2013). These religious processions are considered the forerunners of the "Lubenas," the street processions that are held for nine straight nights during the Misa de Gallo. In these events, white paper lanterns were shaped into various images like crosses, stars, fish, angels and sheep, and used to illuminate the images of patron saints (Orejas, 2012).

The art of parol making is credited to Francisco Estanislao, who in 1908 was said to have made the first parol utilizing a five-point star design. In Estanislao's time, electricity had not yet come to Bacolor (Tapnio, 2018). Townsfolk used parols to light their path going to the traditional dawn masses or Misa de Gallo. In making the parol, Estanislao employed bamboo sticks for the frame and papel de japon for the finishing. Carbide, locally known as kalburo, was used to illuminate the lantern (Cultural Center of the Philippines, 2017).

According to De la Cruz (2013), the first parol festival occurred in 1930. The term "Ligligan Parul" or lantern showdown was an exhibition of how electricity, which was new at that time, could power the light bulbs of the parols for the entire night. Sometime in the 1950's, parol makers started to leave behind the classic five-pointed star made with bamboo and papel de japon and began employing various designs such as psychedelic kaleidoscopes, stained glass windows, prismatic pinwheels, oversized snowflakes, and batik textiles. Then in the 1960's, parol makers ventured into the commercialization of their lanterns and a cottage industry was born (De la Cruz, 2013).

Through the years technological advances have enabled parol makers to create more complex designs and lighting sequences. Pampanga lanterns have become a staple of exhibitions, and Kapampangan lantern makers are regularly commissioned to create special lanterns to commemorate events not only in the Philippines but in different parts of the world (Arvin B. Quiwa, personal communication, August 15, 2015).

(4) DESIGN OF THE PAROL

The traditional design of a parol is a star-shaped framework constructed with bamboo sticks. To complete the parol, crepe paper or other decorative material like cellophane is pasted over the structure after which two tails made of strips of crepe paper are attached to two points of the star. According to Rolando S. Quiambao, a noted lantern designer and maker in Pampanga, the traditional lantern has four main parts. The "tambor," a term derived from the Filipino word tambol or drum, is the center of the lantern. Around the tambor is the "siku- siku" or the main star. Siku is the Kapampangan word for elbow. The "palimbon" or procession surrounds the "siku-siku." Finally, there is the outer layer or the "puntetas" which is derived from the word punta or edge (personal communication, August 15, 2020). Figure 1 shows Quiambao with one of his designs.

Figure 1
Ronald S. Quiambao, noted lantern designer and maker in Pampanga, with one of his designs.

In the early days of giant lantern making, artisans welded steel frames to set up the framework of the parol design. Cardboard was used to line the frame, and then thousands of incandescent light bulbs were installed and wired. In those years, individual switches were used to control the lights, and these were turned on and off in time with the music. Colored paper was then used instead of papel de japon or crepe paper. Later on, rotors, large steel barrels powered by electricity, replaced the hand-controlled switches in producing the desired effect of dancing lights. This was made possible by putting masking tape on the rotor to establish the lighting sequence. The light bulbs were connected to the rotor through hairpins that were attached to the ends of the wires (City of San Fernando, Pampanga, n.d.). Rotors are made of aluminum sheets rolled into a barrel shape which varies in size from 3 inches to 5 meters long depending on the desired light patterns. Operators turn a steering wheel that is welded to the rotor to make half or full turns to achieve the flashing of lights in the course of a performance. After long years of using the rotors and incandescent bulbs, LED electronic lights became the norm, and to produce the dancing light effect, some lantern makers have resorted to the use of sequencers. Designed by Alyosha Ezra Mallari and his team of engineers, sequencers automate the lighting system and eliminate the need for a rotor operator. The desired sequence of lights is written in a program and uploaded to the sequencer which is a machine the size of the CPU of a computer. However, lantern makers still prefer the use of rotors for artistic purposes and to uphold tradition (Orejas, 2017).

Evidently, the making of giant parols is a complex process. However, Quiambao revealed that lantern designers do not actually undergo formal training. Instead, they become apprentices and learn from practice. The current crop of lantern designers in Pampanga started their career in the giant lantern making industry and then ventured into the commercial production of lanterns (personal communication, August 15, 2015).

The beautiful giant lanterns of Pampanga are also the byproduct of the competitive and collaborative spirit of lantern makers from the barangays and towns participating in the lantern festival called "Ligligan Parul" or Giant Lantern Festival held every December of each year. One of the major competitors in this competition, Arvin B. Quiwa, detailed the intricate and modern process of producing a giant lantern. In the recent past, the first step for lantern makers was the design conceptualization where current trends were researched and used as references. Then, a plywood prototype was created for the layout of the frame which was either a 5- or 10-point star. Today, this process has been replaced with the use of photo editing and drafting software that generates a tarpaulin printout which serves as the pattern for creating the welded metal framework. Afterwards, the LED lighting is installed. Alvin Quiwa added that contemporary parols now use a white finishing instead of a colored one, and it is the sequence of LED lighting effects which produces the parol design. He further mentioned that in color selection, contrasting colors are utilized to make the geometric shapes in the pattern distinct from each other and that as many as 15,000 bulbs are sometimes used to realize a particular design (personal communication, August 15, 2020). Figure 2 shows photos of Arvin B. Quiwa and his brother, Eric B. Quiwa, also a giant lantern artist, with lanterns they designed.

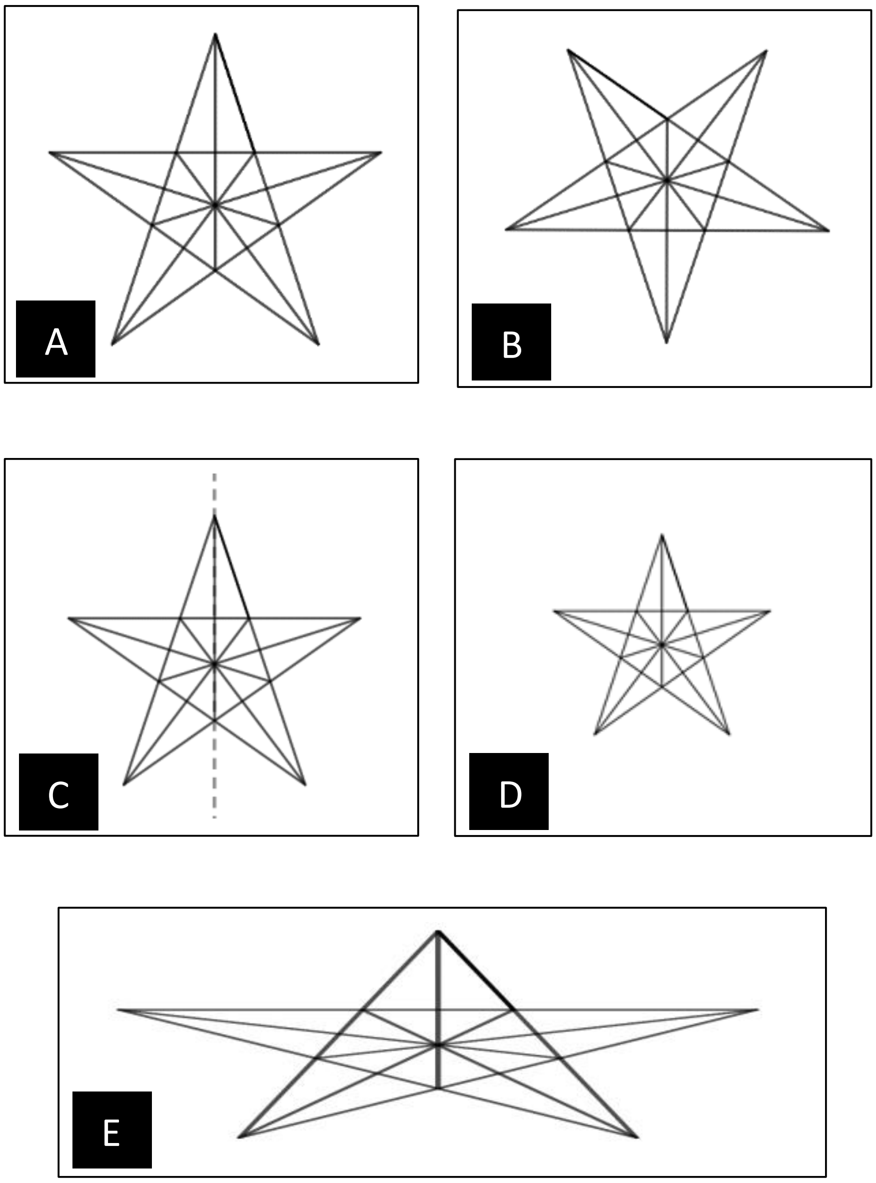
Figure 2
Renowned Parol Artists from Pampanga

Note: (A) Arvin B. Quiwa and (B) Eric B. Quiwa (rightmost in photo) come from a family of renowned giant lantern artists. The giant lantern in the photo of Eric B. Quiwa will be displayed in Lapulapu City as part of its quincentennial commemoration.

(5) SYMMETRIES OF THE PAMPANGA PAROL

To investigate the symmetries of the Pampanga parol, we consider it as a two-dimensional pattern. We alter patterns by applying *transformations*. Figure 3 shows how a star pattern is transformed by applying certain transformations: a rotation, a reflection, a contraction (resizing), and a stretch. We call the transformed pattern the original pattern's *image under the transformation*.

Figure 3
A two-dimensional star pattern and its images under certain transformations



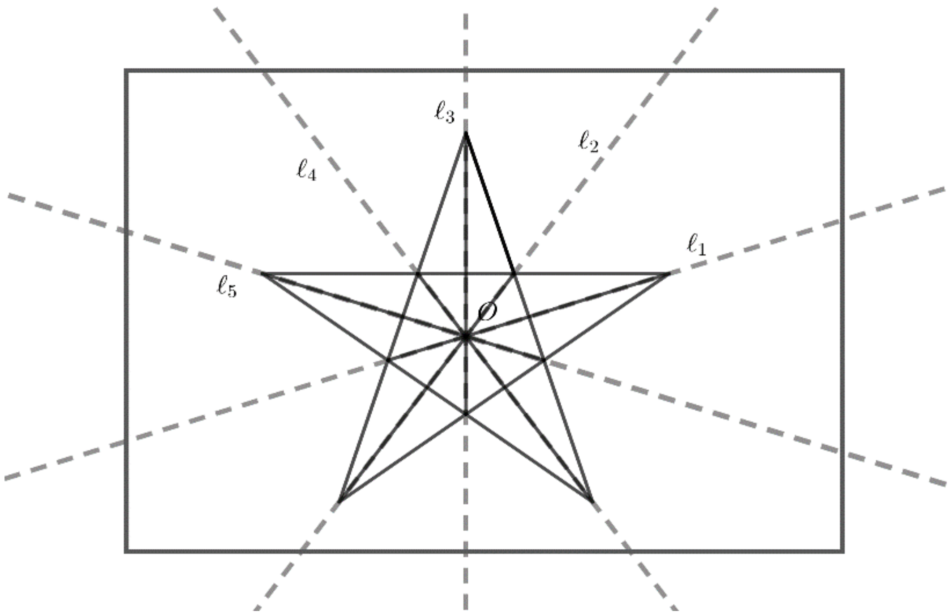
Note: The various transformations of a two-dimensional star pattern are shown above. (A) the original star pattern and its images after (B) a 36-degree rotation about the center of the pattern, (C) a reflection along a vertical line passing through the center of the pattern, (D) a contraction by 75%, and (E) a horizontal stretch.

In the study of symmetry, we are interested in a special type of transformation called *isometries.* Isometries preserve lengths, and consequently, do not alter the shape and size of an object. This means that the image of a pattern under an isometry is congruent to the original pattern. Rotations and reflections are isometries, while contractions and stretches are not, as seen in Figure 3. An isometry that does not alter a pattern is said to be a *symmetry of the pattern*. Since the image of a pattern under a symmetry is the same as the original pattern, we also say that a symmetry of a pattern *fixes* the pattern. Among the five transformations shown in Figure 3(b)-(e), we see that only the reflection in Figure 3(c) is a symmetry of the star pattern.

Mathematical symmetry classifies an uncolored parol pattern as a *discrete finite pattern* (Grünbaum & Shephard, 1987). Such patterns can only have rotation and reflection symmetries (Conway et al., 2008). We denote a rotation by $Rot(O,\theta)$, where O is the center of the rotation and θ is the measure of its angle of rotation. By convention, a positive θ is measured in the counterclockwise direction. We further assume for convenience that θ is in degrees and $\theta \in [0,360)$. (If $\theta \notin [0,360)$, we convert it to r, where $r = \theta - \lfloor \theta/360 \rfloor \cdot 360$ and $\lfloor \cdot \rfloor$ denotes the floor or greatest integer function.) Meanwhile, a reflection is identified with its axis of reflection ℓ, so we write a reflection as $Ref(l)$. Given a finite pattern centered at O, all its rotation symmetries are of the form $Rot(O,\theta)$, with θ rational, while all its reflection symmetries are of the form $Ref(l)$ where ℓ passes through O (Conway et al., 2008). Refer to the star pattern shown in Figure 4. It has five rotation symmetries: $Rot(O,0)$, $Rot(O,72)$, $Rot(O,144)$, $Rot(O,216)$ and $Rot(O,288)$. It also has five reflection symmetries which are $Ref(l_1)$, $Ref(l_2)$, $Ref(l_3)$, $Ref(l_4)$ and $Ref(l_5)$, where l_1, l_2, l_3, l_4 and l_5 are lines passing through O and inclined at 18°, 54°, 90°, 126°, and 162°, respectively, from the right side of the horizontal line passing through *O.*

We define the *product* $t_1 t_2$ of two transformations t_1 and t_2 as the transformation obtained when we apply t_2 followed by t_1. Products of transformations are not commutative since $t_1 t_2$ is not always the same as $t_2 t_1$. Table 1 gives the products of rotations and reflections that are of interest to us.

Figure 4
Symmetries of a discrete, finite star pattern



The *identity transformation*, denoted by e, is the transformation that fixes any pattern. We identify e with $Rot(O, 0)$ since a rotation by 0° does not alter any pattern. In this sense, we say $Rot(O, 0°)$ is the *trivial rotation*. Observe that if t is any transformation, $te = et = t$. The *order* of a transformation t is the smallest positive integer n such that $t^n = e$, if it exists. It is easy to see that e is of order 1, while a reflection is of order 2. On the other hand, if $\theta = p/q$, where p and q are both nonzero integers such that $gcd(p, q) = 1$, the order of $Rot(O, \theta)$ is $lcm(p, 360q)/p$. If t is a transformation, its *inverse* t^{-1} is the transformation that satisfies $tt^{-1} = t^{-1}t = e$. If t is a rotation of order n, then $t^{-1} = t^{n-1}$. Meanwhile, if t is a reflection, then $t^{-1} = t$. Furthermore, it is easy to show that if $t_1 = Rot(O, \theta)$ and $t_2 = Ref(l)$ such that O is on ℓ, then $t_2t_1 = t_1^{-1}t_2$.

Table 1
Products of some rotations and reflections

t_1	t_2	$t_1 t_2$
$Rot(O, \theta_1)$	$Rot(O, \theta_2)$	$Rot(O, \theta_1 + \theta_2)$, this product is commutative
$Rot(O, \theta)$	$Ref(l)$, ℓ passes through O	$Ref(m)$, where m passes through O and the angle between ℓ and m, measured from ℓ to m is $\theta/2$
$Ref(l)$	$Ref(m)$, where l and m intersect at O	$Rot(O, 2\theta)$, where θ is the angle between l and m, measured in the direction from m to ℓ.

In general, given the set of symmetries of a pattern, the product of two symmetries is also a symmetry, the identity transformation is a symmetry, and the inverse of a symmetry is also a symmetry. Thus, the set of symmetries of a pattern has the structure of a mathematical group, and we call this set the *symmetry group* of the pattern. If G is the symmetry group of a finite discrete pattern, then G is either a finite cyclic group C_n or a finite dihedral group D_n (Grünbaum & Shephard, 1987). In the first case, $G = \{e, a, a^2, \dots, a^{n-1}\}$, while in the latter case, $G = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$, where a is a rotation of order n with the smallest angle of rotation, and b is any reflection in G. Since every element of G can be written as a product of a power of a and/or b, we say that a and/or b *generate/s* G. We write this as $G = \langle a \rangle$ when $G \cong C_n$ and as $G = \langle a, b \rangle$ when $G \cong D_n$.

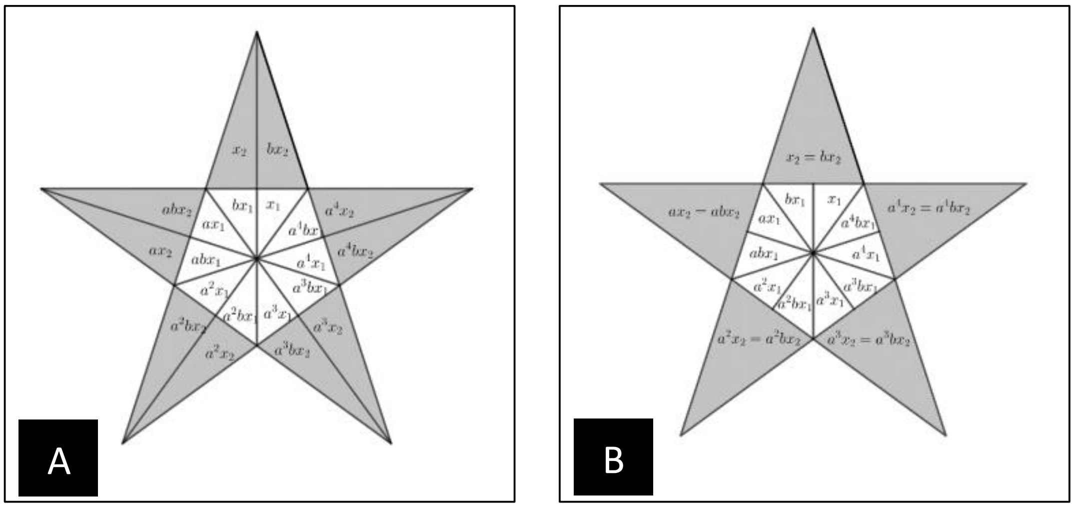
Consider the symmetries of the star pattern centered at O as shown in Figure 4. Then, $G = \langle a, b \rangle$, where $a = Rot(O, 72)$ and $b = Ref(l_3)$. (We can use any of the reflections to generate G, but we use the vertical line passing through O for convenience). Since a is of order $n = 5$, we have $G \cong D_5$.

Since the uncolored Pampanga parol is normally centered on a five-pointed star, its symmetry group is usually C_n or D_n, where n is some multiple of 5 (City of San Fernando Pampanga Tourism Office, 2015; "Amazing Giant Parol Festival in Pampanga", n.d.). Though this seems to be the rule, there are exceptions such as the uncolored parol in Figure 2(B), whose symmetry group is C_8. Hence, we see that a few of the newer generation parol artists are beginning to move away from the tradition of basing their designs on the five-pointed star.

The Pampanga parol is composed of different shapes with specific positions in two-dimensional space. We think of these shapes, together with their position in space, as the *tiles* that make up the pattern. In Figure 5(a), the non-overlapping triangles are the tiles of the pattern. Now suppose X is the set of tiles of some uncolored finite discrete pattern with symmetry group G. If $g \in G$ and x is a tile in X, we denote by gx the image of x under the symmetry g. Since g is in the symmetry group of X, gx is also a tile in X. In fact, $G(X) = \{gx \mid g \in G, x \in X\} = X$. Hence, G *acts* on X. The stabilizer of x in G, denoted $Stab_G x$, is the set of all elements in G that fix x, or $Stab_G(x) = \{gx = x\}$. The *orbit* of a tile x under the action of G, written Gx, is the set of all images of x under all symmetries in G, or $Gx = \{gx \mid g \in G\}$. If x_1 and x_2 are tiles in X, then Gx_1 and Gx_2 are either equal or disjoint. Furthermore, the union of all the orbits is X. Hence, the collection of unique orbits is a partition of X. We observe this in Figure 5(a). Let X be the set of tiles in the star pattern and let x_1 and x_2 be as shown in the figure. We see that X has two orbits $X_1 = Gx_1$ and $X_2 = Gx_2$ under the action of G and $X_1 \cup X_2 = X$.

We say that two sets have a *one-to-one correspondence* if we can define a function between the two sets such that each element of one set is associated with one and only one element of the other set. It is easy to see that two finite sets have a one-to-one correspondence if and only if they have the same number of elements. Hence, if the symmetry group G is finite, then G and the orbit Gx of x under the action of G have a one-to-one correspondence if G and Gx have the same number of elements. The two-star patterns in Figure 5 have the same symmetry group G. Each orbit of the star pattern in Figure 5(a) has a one-to-one correspondence with G. However, this is not the case for the star pattern in Figure 5(b) since G has 10 elements while Gx_2 has only 5 elements.

Figure 5
Two-star Patterns with the Same Symmetry Group



Note: (A) A star pattern with two orbits under its symmetry group. The white tiles belong to the first orbit, while the grey tiles belong to the second orbit. Each orbit has a one-to-one correspondence with the pattern's symmetry group (B) A star pattern with two orbits under its symmetry group. The first orbit consists of the ten white tiles and has a one-to-one correspondence with the pattern's symmetry group. The second orbit consists of the five gray tiles; it does not have a one-to-one correspondence with the pattern's symmetry group.

(6) COLOR SYMMETRIES OF THE PAMPANGA PAROL

Consider the problem of coloring the Pampanga parol so that the colored pattern also exhibits some form of symmetry. We call the colored version of an uncolored pattern X a *coloring of X*. *Coset coloring* is a mathematical method used to obtain symmetric colorings of a pattern (Felix, 2011).

To understand coset coloring, we need some additional concepts from abstract algebra. Let G be the symmetry group of X. A *subgroup* H of G is a subset of G that also has a group structure. Recall that the symmetry group of a discrete finite pattern is $G = \langle a \rangle \cong C_n$ or $G = \langle a, b \rangle \cong D_n$, where a is a rotation of order n and b is a reflection. If $G = \langle a \rangle \cong C_n$, the subgroups of G are of the form $H = \langle a^r \rangle \cong C_{n/r}$, where r is a divisor of n. If $G = \langle a, b \rangle \cong D_n$, the subgroups of G are either $H = \langle a^r \rangle \cong C_{n/r}$ or $H = \langle a^r, a^s b \rangle \cong D_{n/r}$, where *r* is a divisor of *n* and $s \in \{0,1,2, \dots, r-1\}$.

A *left coset* of a subgroup H of G is a set $gH = \{ h \in H\}$, where g is some element of G. Meanwhile, a *right coset* of H is $Hg = \{ h \in H\}$, where g is some element of G. It is possible for $g_1 H = g_2 H$ or $Hg_1 = Hg_2$ even if $g_1 \neq g_2$. Given a subgroup H of G, the number of unique left cosets of H is equal to the number of unique right cosets. This number is known as the *index* of H in G, and we denote this by $[G: H]$. When G is finite, $[G: H] = |G|/|H|$, where $|\cdot|$ indicates the number of elements in the group. Generally, the left cosets of H are different from its right cosets. In case they coincide, we say that H is a *normal subgroup* of G. If $G = \langle a \rangle \cong C_n$, then all subgroups of H are normal. If $G = \langle a, b \rangle \cong D_n$, then $H = \langle a^r \rangle \cong C_{n/r}$ is always normal in G. However, $H = \langle a^r, a^s b \rangle \cong D_{n/r}$ is normal in G if and only if H is one of the following subgroups: $H = G$, $H = \{e\}$ or $H = \langle a^2, a^s b \rangle$. Clearly, the last case can happen only if *n* is even.

The set of left or right cosets of H is a partition of G. Under certain conditions, this partition of G gives rise to a partition of X. In general, we obtain coset colorings by associating a color to each set in the partition of X. However, the properties of colorings using left cosets are different from those of colorings using right cosets. For example, colorings using left cosets are said to be *perfect* in the sense that all elements of G induce a permutation of colors. That is, if $g \in G$ and C is the set of all tiles of a particular color, then all the tiles in gC have the same color. In fact, a coloring is perfect if and only if the coloring is a left coset coloring (Evidente, 2012; Loquias & Frettlöh, 2017; Junio & Walo, 2019). Colorings using right cosets have less symmetry, which sometimes lead to less typical and more interesting designs. The framework for coloring using left cosets when X has only one orbit under the action of G is discussed by Felix (2011). His students and their collaborators have extended this coloring framework to accommodate multiple orbits (Loquias & Frettlöh, 2017; Junio & Walo, 2019). We apply the extended framework since the set of tiles of a Pampanga parol pattern has multiple orbits under the action of its symmetry group. We also attribute to Felix the fundamentals for the right coset coloring method presented in this paper, which he discussed in numerous lectures. We are not aware of any published work using right cosets to obtain symmetric colorings.

(6.1) Left Coset Colorings

Fix $x \in X$. If gH is a left coset of H, then $gHx = \{\ h \in H\}$ is the set of images of x under the elements of gH. The next theorem gives the basis for left coset colorings. We do not discuss the proof here, but the interested reader may refer to (Evidente, 2012) and (Loquias & Frettlöh, 2017).

Theorem 1. Suppose X is the set of tiles in an uncolored finite discrete pattern with symmetry group G such that X has m orbits $X_1, X_2, \dots, X_m$ under the action of G. Fix any $x_i \in X_i$.

(i) *Coloring Method 1*: Let H_i be a subgroup of G such that H_i contains $Stab_G x_i$. Then $\cup_{i=1}^{m} \{g \in G\}$ is a partition of X, and we can color X symmetrically by assigning a color to each set in the partition.

(ii) *Coloring Method 2*: Let H be a subgroup of G such that H contains $Stab_G x_i$ for all $i \in \{1,2,\dots,m\}$. Then $\{g \in G\}$ is a partition of X, and we can color X symmetrically by assigning a unique color to each set in the partition.

We apply Theorem 1 to obtain a left coset coloring of a pattern by following this procedure:

1. Determine the symmetry group G of the pattern.
2. Let X be the set of tiles in the pattern. Find the orbits $X_1, X_2, \dots, X_m$ of X under the action of G. Choose an element x_i from each orbit X_i, $i \in \{1,2,\dots,m\}$
3. For Coloring Method 1: For each $i \in \{1,2,\dots,m\}$
   a. Select a subgroup H_i such that H_i contains the stabilizer of x_i.
   b. Determine the left cosets of H_i.
   c. For each left coset gH_i of H_i, get the set of images $gH_i x_i$ and assign the same color to the tiles in this set.
4. For Coloring Method 2:
   a. Select a subgroup H that contains the stabilizer of x_i for all $i \in \{1,2,\dots,m\}$.
   b. Determine the left cosets of H.
   c. For each left coset gH
      i. For each $i \in \{1,2,\dots,m\}$, get the set of images gHx_i.
      ii. Assign the same color to the tiles in the set $\cup_{i=1}^{m} gHx_i$.

Example. Let us color the star pattern in Figure 5(a) using left coset-method 2 following the steps outlined above. The symmetry group of the star is $G = \langle a, b \rangle \cong D_5$, where $a = Rot(O, 72)$, $b = Ref(l)$, O is the center of the star and l is the vertical line passing through O. Let X be the set of tiles in the star pattern and select x_1 and x_2 as shown in Figure 5(a). Now X has two orbits $X_1 = Gx_1$ and $X_2 = Gx_2$ under the action of G. The subgroups of G are of the form $H = \langle a^r \rangle \cong C_{5/r}$ or $H = \langle a^r, a^s b \rangle \cong D_{5/r}$, where $r \in \{1,5\}$ and $s \in \{0,1,2, \ldots, r-1\}$. Table 2 lists all subgroups H of G.

Table 2

Subgroups of the symmetry group $G = \langle a, b \rangle \cong D_5$

H	**Elements**	$\lvert H \rvert$	**Index**	**Normal**
$H = \langle e \rangle$	e	1	10	Yes
$H = \langle a \rangle$	e, a, a^2, a^3, a^4	5	2	Yes
$H = \langle a^s b \rangle$, $s \in \{0,1,2,3,4\}$	$e, a^s b$	2	5	No
$H = \langle a, b \rangle$	$e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a$	10	1	Yes

Note: Subgroups of the symmetry group $G = \langle a, b \rangle \cong D_5$ of the star pattern, where $a = Rot(O, 72), b = Ref(l)$, O is the center of the star and l is the vertical line passing through O.

Note that $Stab_G x_1 = Stab_G x_2 = \{e\}$. Hence, we can select any subgroup H listed in Table 2. Let us choose $H = \langle a \rangle$. The left cosets of H are $H = \{e, a, a^2, a^3, a^4\}$ and $bH = \{b, ab, a^2b, a^3b, a^4b\}$. We compute the images of x_1 and x_2 under the elements of H and obtain $Hx_1 = \{x_1, ax_1, a^2x_1, a^3x_1, a^4x_1\}$ and $Hx_2 = \{x_2, ax_2, a^2x_2, a^3x_2, a^4x_2\}$. Meanwhile, the images of x_1 and x_2 under the elements of bH are $bHx_1 = \{bx_1, abx_1, a^2bx_1, a^3bx_1, a^4bx_1\}$ and $bHx_2 = \{bx_2, abx_2, a^3bx_2, a^4bx_2\}$, respectively. We assign the color white to $Hx_1 \cup Hx_2$ and the color blue to $bHx_1 \cup bHx_2$ and obtain the coloring of the star pattern shown in Figure 6(a). Since this is a left coset coloring, observe that the generators of G permute the colors of the coloring: a sends white to white and blue to blue, while b sends white to blue and blue to white. Since the generators of G permute the colors, it follows that all elements of G also permute the colors.

(6.2) Right Coset Colorings

Fix $x \in X$. If Hg is a right coset of H, then $Hgx = \{ h \in H\}$ is the set of images of x under the elements of Hg. We obtain right coset colorings by applying the next theorem. Again, we do not prove the theorem here, but it is easy to show using basic definitions and theorems from abstract algebra.

Theorem 2. Suppose X is an uncolored pattern with symmetry group G such that X has m orbits $X_1, X_2, \dots, X_m$ under the action of G and each orbit has a one-to-one correspondence with the elements of G. Fix $x_i \in X_i$.

(i) *Coloring Method 1*: Let H_i be a subgroup of G. Then $\cup_{i=1}^{m} \{g \in G\}$ is a partition of X, and we can color X symmetrically by assigning a unique color to each set in the partition.

(ii) *Coloring Method 2*: Let H be a subgroup of G. Then $\{g \in G\}$ is a partition of X, and we can color X symmetrically by assigning a unique color to each set in the partition.

We make some observations about right coset colorings. If all subgroups of G are normal, then the right cosets of any subgroup coincide with its left cosets. Hence, opting for right coset coloring only makes sense when G has at least one subgroup that is not normal. Furthermore, right coset coloring assumes that all orbits of X have a one-to-one correspondence with G. To work around this limitation, we only use method 1 when not all orbits have a one-to-one correspondence with G. For orbits that do not have a one-to-one correspondence with G, we use a normal subgroup of G containing the stabilizer of an element in that orbit to color that orbit. (In effect, we are using left coset coloring for that orbit.)

Thus, we can obtain a right coset coloring of a pattern by applying Theorem 2 using the following procedure:

1. Determine the symmetry group G of the pattern.
2. Let X be the set of tiles in the pattern. Find the orbits $X_1, X_2, \dots, X_m$ of X under the action of G. Choose an element x_i from each orbit X_i, $i \in \{1,2,\dots,m\}$.
3. For Coloring Method 1: For each $i \in \{1,2,\dots,m\}$.
   a. Select a subgroup H_i. If X_i does not have a one-to-one correspondence with G, choose a normal subgroup containing the stabilizer of x_i.

b. Determine the right cosets of H_i.
c. For each right coset $H_i g$ of H_i, get the set of images $H_i g x_i$ and assign the same color to the tiles in this set.

4. For Coloring Method 2:
    a. Select a subgroup H.
    b. Determine the right cosets of H.
    c. For each right coset Hg
        i. For each $i \in \{1,2,\dots,m\}$, get the set of images Hgx_i.
        ii. Assign the same color to the tiles in the set $\cup_{i=1}^{m} Hgx_i$.

Example. We color the star pattern in Figure 5(b) using the procedure above. The symmetry group of the star pattern is $G = \langle a, b\rangle \cong D_5$, where $a = Rot(O, 72)$, $b = Ref(l)$, O is the center of the star, and l is the vertical line passing through O. If X is the set of tiles in the star pattern, then the orbits of X under the action of G are given by $X_1 = Gx_1$ and $X_2 = Gx_2$. Notice that X_1 has a one-to-one correspondence with G, but X_2 does not. Thus, we can only use method 1. We can use any subgroup to color X_1. Choose $H_1 = \langle b\rangle$, with right cosets $H_1, H_1a, H_1a^2, H_1a^3, H_1a^4$. We need to use a normal subgroup of G that contains the stabilizer of x_2 to color X_2. Since $Stab_G x_2 = \{e, b\}$, our only option is to use $H_2 = \langle a, b\rangle$. Since $H_2 = G$, the only coset of H_2 is itself.

The sets of images of x_1 under the elements of the cosets of H_1 are

$$H_1x_1 = \{x_1, bx_1\} \qquad H_1ax_1 = \{ax_1, a^4bx_1\} \qquad H_1a^2x_1 = \{a^2x_1, a^3bx_1\}$$

$$H_1a^3x_1 = \{a^3x_1, a^2bx_1\} \qquad H_1a^4x_1 = \{a^4x_1, abx_1\}$$

Meanwhile the set of images of x_2 under the elements of H_2 is

$$H_2x_2 = \{x_2, ax_2, a^2x_2, a^3x_2, a^4x_2, bx_2, abx_2, a^2bx_2, a^3bx_2, a^4bx_2\}$$

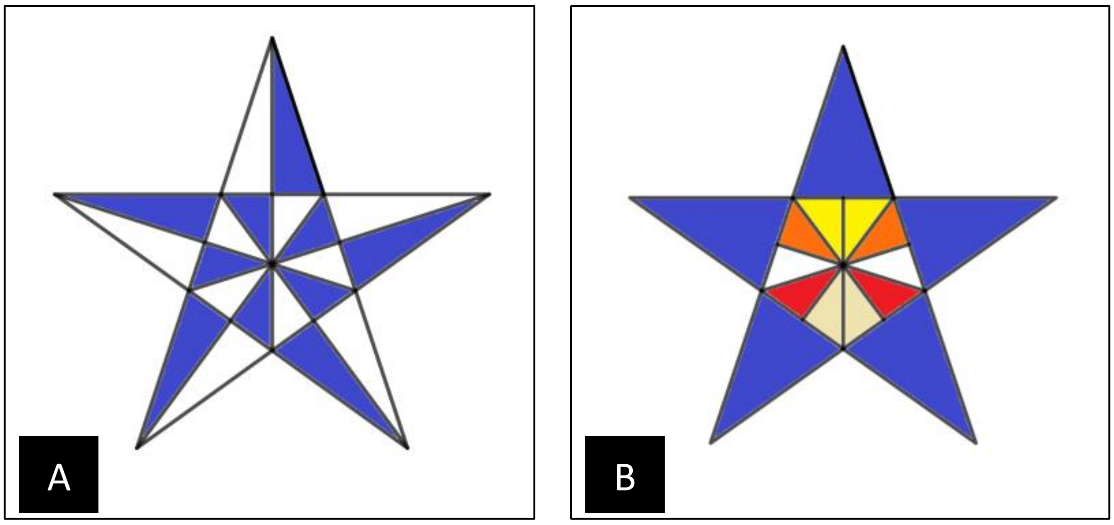
But some of the elements in H_2x_2 are equal, and we get

$$H_2x_2 = \{x_2, ax_2, a^2x_2, a^3x_2, a^4x_2\}$$

If we assign the colors yellow, orange, red, beige, white, and blue to $H_1x_1, H_1ax_1, H_1a^2x_1, H_1a^3x_1, H_1a^4x_1$, and H_2x_2, respectively, we obtain the coloring shown in Figure 6(b).

Figure 6

A left coset-method 2 coloring of the star pattern (A), and a right coset-method 1 coloring of the modified star pattern (B)

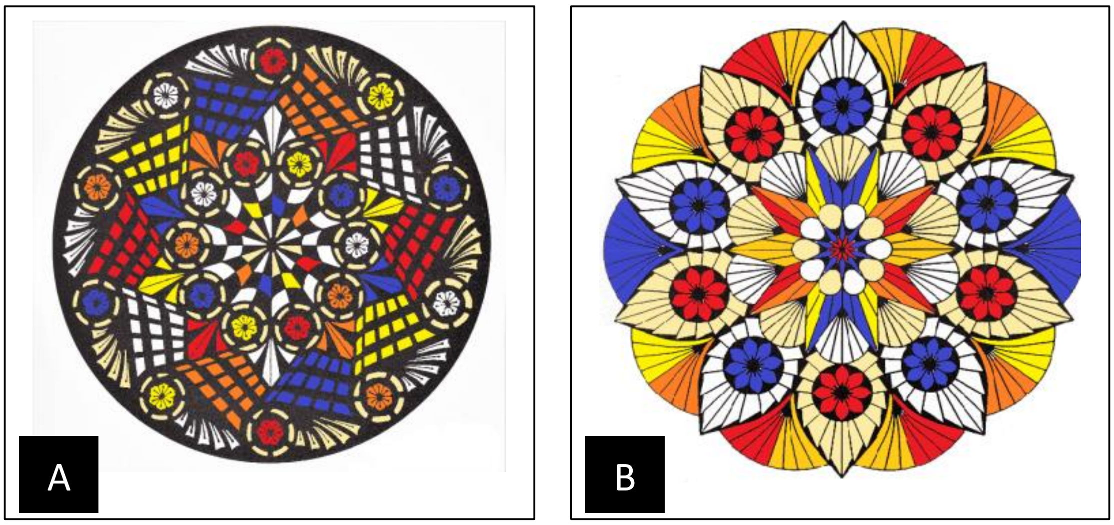


(6.3) Coset Colorings of the Pampanga Giant Lantern

If we examine the recent designs of giant lanterns by Pampanga's artists, we see that the colorings are usually based on left coset colorings, where the full symmetry group or an index 2 subgroup of the symmetry group is used. This is the case for the colorings in Figure 1 and Figure 2(b). This feature is also present in the majority of the giant lanterns displayed during the Giant Lantern Festival ("Amazing Giant Parol Festival in Pampanga", n.d.). Observe that the elements of the symmetry group permute the colors in each orbit, indicating that the coloring can be obtained using left cosets. Furthermore, there are at most two colors in each orbit, so either an index 1 or index 2 subgroup of the symmetry group was used.

We now derive some symmetrical colorings of the parol that are dissimilar to existing designs by using left coset colorings with subgroups of higher index or by using right coset colorings. We applied the coset coloring method to the giant lantern patterns in the Parul Sampernandu Coloring Book (City of San Fernando Pampanga Tourism Office, 2015). Although not all these designs have been displayed in the Giant Lantern Festival, these were designed by some of Pampanga's renowned artists. The symmetry groups of most giant lantern patterns in the coloring book are of type D_{10}, while some are of type C_{10} or D_5. The patterns are very intricate such that the set of tiles usually have more than 20 orbits under the symmetry group. Since there are too many orbits, we opted to use the same color in distinct orbits at times. Figure 7 shows two patterns taken from the coloring book. The symmetry group of the pattern shown in Figure 7(a) is $G = \langle a \rangle \cong C_{10}$. The coloring is obtained by left coset coloring-method 2 using the subgroups $\langle a \rangle$, $\langle a^2 \rangle$ and $\langle a^5 \rangle$. Figure 7(b) shows a coloring of a parol pattern with symmetry group $G = \langle a, b \rangle \cong D_{10}$. The pattern is colored using right coset-method 2 since not all orbits have a one-to-one correspondence with the symmetry group. The subgroups $\langle a, b \rangle$, $\langle a^2, ab \rangle$ and $\langle a^5, b \rangle$ are used for the coloring.

Figure 7
Examples of a left coset coloring (A) and a right coset coloring (B) of the Pampanga Parol



(7) CONCLUSION AND RECOMMENDATIONS

This paper examines the symmetries and color symmetries of Pampanga's giant lanterns. The symmetry group of the uncolored designs are either finite cyclic or a finite dihedral where the group of rotations is generated by a rotation or order n about the center, and where n is usually a multiple of 5. The existence of designs where n is not a multiple of 5 reveals that some artists are already departing from the tradition of centering their designs on the five-pointed star. Color symmetry analysis reveals that colored designs by parol artists match left coset colorings where the subgroups of the symmetry group used for the coloring is either index 1 or 2. We show how to obtain a different variety of colorings by using higher index subgroups with left coset coloring or by using right coset coloring.

Left or right coset colorings are just one of many methods that allow us to come up with symmetric colorings. Other methods include employing an algorithm that uses double cosets or exploring colorings that may be obtained from different coloring methods that specify the level of symmetry attained: perfect, semiperfect, chirally perfect. One may also attempt to find a more general approach to address the limitation encountered in right coset colorings when there is no one-to-one correspondence between the symmetry group and the orbits of the tiles under the symmetry group. In fact, those interested in the theory of color symmetry may analyze the structural properties of right coset colorings and double coset colorings.

We hope that this contribution encourages mathematicians to do research on math art, artists to explore the ways math can be used to generate art, and teachers to use math art for instruction. We also hope this initiates more collaboration between mathematicians and artists, especially those of Central Luzon. This paper is just a glimpse into the world of math art. Indeed, there are still many areas to be explored in this fascinating field.

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