Problem Solving Strategies of High School Students on Non-Routine Problems: A Case Study

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Abstract

The main purpose of this study was to investigate how well certain students in a university high school solve non-routine problems. These problem situations required the use of their conceptual understanding of mathematics and their procedural knowledge of the algorithm involved in the solution. Results of analysis of students' solutions showed that each student employed at least four problem-solving strategies. Seven out of the eight possible problem solving strategies were used at least once to solve the twelve non-routine problems. When given the opportunity to use any problem solving strategy, the students solved non-routine problems even without prior instruction. The most frequently used strategy was "Making a Model or Diagram." After evaluating the students' problem solving skills using the Oregon Mathematics Problem Solving Rubric, results showed that of the five students, three had proficient level of conceptual understanding and procedural knowledge, one was a transitional problem solver from apprentice to proficient level of performance, and one was an apprentice problem solver. Those who performed well were also proficient in the use of solution strategies whereas the transitional problem solvers were either novice to apprentice or apprentice to proficient in their use of solution strategies.

Introduction

The results of the National Achievement Test (NAT) conducted among secondary students in a university high school in the academic year 2009-2010 showed a low performance with a mean percentage score of only 60.24 in the Mathematics sub-test. Since most of the items in the sub-test are application problems, this reveals the weak problem solving skills of the students. Considering that these students belong to the high ability group in terms of rank in the entrance examination, this low performance in NAT is very alarming. An investigation about their problem solving abilities was therefore, deemed necessary.

Problem solving is regarded as one of the primary skills that students must take with them when they leave the classrooms and enter the real world (Krulik and Rudnick 1996). Aside from developing the critical and analytical thinking skills of the students, it promotes conceptual understanding and meaningful learning in mathematics. Limjap (1996) reported that as students are given the opportunity to reflect on their experiences when they confront problem situations, they learn to construct their own ways of reasoning in mathematics. She further added that the students come to understand their own learning process and are able to deal with problem situations which facilitate their understanding of the mathematics concepts. However, math problems may be classified either as routine problems that directly apply formulas learned in class or non-routine problems that require the application of various mathematical concepts. Non-routine problems help develop critical and creative thinking among students. For Candelaria and Limjap (2002), the development of critical thinking skills essential for problem solving does not necessarily require direct instruction. Students may acquire the skill as they interact with their environment in the school and at home, thus honing their creative skills as well.

Problem Solving Heuristics

According to Candelaria and Limjap (2002), a problem solving process is referred to as a heuristic. For Polya (1945), problem solving heuristics are not clear-cut rules in coming up with the correct answers, rather they are possible solutions for certain problems. Polya (1957) states that successful problem solving involves four steps:
a) understanding the problem; b) selecting strategy; c) solving the problem; and
d) looking back. Students may demonstrate problem solving strategies with clear and good reasoning that leads to a successful resolution of the problem. Krulik and Rudnick (1996) identified eight strategies (along with their definitions) that are applicable to mathematical problem solving at the secondary level, namely:

1. Computing or Simplifying (CS) includes straightforward application of arithmetic rules, order of operations, and other procedures.

2. Using a Formula (F) involves substituting values into a formula or selecting the proper formula to use.

3. Making a Model or Diagram (MD) includes use of objects, drawings, acting out, or writing an equation.

4. Making a Table, Chart, or List (TCL) involves organizing the data by making a table, chart, graph, or list.

5. Guessing, Checking, and Revising (GCR) involve making a reasonable guess, checking the guess, and revising the guess, if necessary.

6. Considering a Simpler Case (SC) includes rewording the problem, using smaller numbers, using a more familiar problem setting, dividing the problem into simpler problems, or working backwards.

7. Eliminating (E) involves eliminating possible solutions based on information presented in the problem or elimination of incorrect answers.

8. Looking for Patterns (LP) involves determining certain common character-ristics that can be generalized and used to solve the problem.

It should be noted that problem solving requires the use of many skills, often in certain combinations, before the problem is solved. Strategies used to solve problems are not explicitly taught by teachers. Circumstantial evidence has suggested that the range of heuristics taught by the teachers are usually limited and the types of problems given in class are usually those found in the textbooks, referred to as routine problems. Students in this study are taught the use of diagram/model heuristics as well as common heuristics like guess and check, listing and working backwards.

In order to enhance the problem solving skills of students, teachers can expose them to unfamiliar problematic situations that challenge their heuristics. This can be accomplished using non-routine problems.

Non-routine problems provide a large room for varied solutions, strategies, and approaches in problem solving. Also, they provide students with a realistic situation in which they will be using higher order thinking skills such as application, synthesis and creation.

A student's understanding of non-routine problems described by the Oregon Department of Education (1991) consists of the following:

a) Conceptual Understanding includes the ability to interpret the problem and select appropriate concepts and information to apply a strategy for solution. Evidence is communicated through making connections between the problem situation, relevant information, appropriate mathematical concepts, and logical/ reasonable responses.

b) Procedural knowledge deals with the student's ability to demonstrate appropriate use of algorithms. Evidence includes the verifying and justifying of a procedure using concrete models, or the modifying of procedures to deal with factors inherent to the problem.

c) Problem Solving Skills and Strategies includes clearly focused good reasoning and insightful thinking that leads to a successful resolution of the problem.

Conceptual understanding is manifested by an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand the importance of a mathematical idea and how it is applied in certain situations. They have organized their knowledge into a coherent whole, enabling them to see the connections of new ideas with what they already know.

Students who have conceptual understanding of mathematical ideas retain those ideas because of the relationships and connections involved. Hence, they find it easy to reconstruct the ideas that they forget. Those who understand only the method oftentimes remember it incorrectly. Those who understand the concept monitor what they remember and try to figure out if it makes sense. They may attempt to explain the method to themselves and correct it if necessary. Evidence of conceptual under-standing is found in the student's ability to verbalize connections among concepts and representations. However, conceptual under-standing does not need to be explicit since most students understand before they can verbalize that understanding.

Devlin (2007) defined conceptual understanding as "the comprehension of mathematical concepts, operations, and relations," which elaborates the question but does not really answer it. He believed that, an important component of mathematics education, is achieving conceptual under-standing. He noted that many mathematical concepts can be understood only after the learner has acquired procedural skill in using the concept. In such cases, learning can take place only by first learning to follow symbolic rules, with understanding emerging later, sometimes considerably later. Devlin (2007) also agreed with practically everyone that procedural skills not eventually accompanied by some form of understanding are brittle and easily lost. He also believed that the need for rule-based skill acquisition before conceptual understanding can develop is in fact the norm for more advanced parts of Mathematics (Calculus and beyond), and he was not convinced on the idea that it is possible to proceed otherwise in all of the more elementary parts of the subject.

Ben-Hur (2006) defined procedural knowledge as knowledge of formal language or symbolic representations. It involves the ability to solve problems through the manipulation of mathematical skills with the help of pencil and paper, calculator, computer, and so forth. In an article by G. Wiesen (2003), procedural knowledge is defined as the type of knowledge someone has and demonstrates through the procedure of doing something.

Problem solving skills and strategies consist of a utilization of the appropriate basic thinking skills and higher order thinking skills needed to solve the problem. This includes flexibility in the choice of the strategy to be employed, openness to try different strategies and self regulation while solving the problem. Since students with this skill are clearly focused on the resolution of the problem, they can recognize the accuracy and reasonableness of the answer.

Purpose of the Study

This study was conducted to investigate the problem solving strategies of five (5) third year students of a university high school. It aimed to answer the following questions:

1. What is the level of performance of the students on non-routine problems in terms of the following:
   a. Conceptual Understanding?
   b. Procedural Knowledge?
   c. Problem Solving Skills and Strategies?

2. What problem solving strategies do students belonging to the same level of performance use in solving non-routine problems?

Methodology

This research utilized the descriptive research design to examine and classify the problem solving strategies of high school students in solving non-routine problems.

The study was conducted in a university high school. Five (5) participants, 3 males and 2 females were chosen among the 124 third year students of the school. They were randomly chosen from all sections of Math classes in the third year level.

Twelve non-routine problems from the sourcebook of Krulik and Rudnick (1996) were used to determine the different strategies of the students. The questions and evaluation criteria were validated by three university professors and two public school teachers. The profiles of the students were collected. The Oregon Mathematics Problem Solving Rubrics, (Oregon Department of Education 1991) were used to evaluate students' conceptual understanding, procedural knowledge and problem solving strategies and skills. The problem solving performance of the students were classified as "proficient", "apprentice", and "novice" (tables 1 to 3).

Full Conceptual Understanding is characterized by the ability of the students to recognize the essence of the problem. They can see the relations of the given information and represent those relations mathematically. They are able to recognize the appropriateness of the answers that they obtained. Different levels of conceptual understanding are classified in table 1.

TABLE 1 Classification of the different levels of conceptual understanding

Full Conceptual Understanding (Proficient)	Partial Conceptual Understanding (Apprentice)	Lack of Conceptual Understanding (Novice)
The student uses all relevant information to solve the problem.	The student extracts the "essence" of the problem, but is unable to use this information to solve the problem.	The student's solution is inconsistent or unrelated to the question.
The student is able to translate the problem into appropriate mathematical language.	The student is only partially able to make connections between/ among the concepts.	The student translates the problem into inappropriate mathematical concepts.
The student's answer is consistent with the question/problem.	The student's solution is not fully related to the question.	The student uses incorrect procedures without understanding the concepts related to the task.
	The student understands one portion of the task, but not the complete task.	

Aside from representing the problem mathematically, students should be able to employ the appropriate steps and procedures in the solution of the problem. Correct algorithm is needed to arrive at the precise answer as indicated in table 2.

TABLE 2 Classification of the use of procedural knowledge

Full Use of Appropriate Procedures (Proficient)	Partial Use of Appropriate Procedures (Apprentice)	Lacks Use of Appropriate Procedures (Novice)
The student uses principles efficiently while justifying the solutions.	The student is not precise in using mathematical terms, principles, or procedures.	The student uses unsuitable methods or simple manipulation of data in his/her attempted solution.
The student uses appropriate mathematical terms and strategies.	The student is unable to carry out a procedure completely.	The student fails to eliminate unsuitable methods or solutions.
The student solves and verifies the problem.	The process the student uses to verify the solution is incorrect.	The student misuses principles or translates the problem into inappropriate procedures.
The student uses mathematical principles and language precisely.		The student fails to verify the solution.

To evaluate the over-all problem solving performance of the students for the 12 problems, a point system was used: 5 points for Proficient, 3 points for Apprentice, and 1 point for Novice. A score of 2 was assigned to work that exceeded criteria for a score of 1, but did not meet criteria for a score of 3. Similarly, a score of 4 was assigned to work that exceeded criteria for a score of 3, but did not meet criteria for a score of 5.

Problem Solving Skills and Strategies are categorized into three (Oregon Department of Education 1991):

TABLE 3 Classification of problem solving skills and strategies

Thorough/Insightful Use of Skills/Strategies (Proficient)	Partial Use of Skills/Strategies (Apprentice)	Limited Skills/Strategies (Novice)
The skills and strategies show some evidence of insightful thinking to explore the problem.	The skills and strategies have some focus, but clarity is limited.	The skills and strategies lack a central focus and the details are sketchy or nor present.
The student's work is clear and focused.	The student applies a strategy which is only partially useful.	The procedures are not recorded (i.e., only the solution is present).
The skills/strategies are appropriate and demonstrate some insightful thinking.	The student starts the problem appropriately, but changes to an incorrect focus.	Strategies are random. The student does not fully explore the problem and look for concepts, patterns or relationships.
The student gives possible extensions or generaliza-tions to the solution or the problem.	The student recognizes the pattern or relationship, but expands it incorrectly.	The student fails to see alternative solutions that the problem requires.

The students' problem solving schema is characterized by their ability to demonstrate deep thinking about the problem. Students should be able to recognize the relationships of the given data and employ the appropriate strategy.

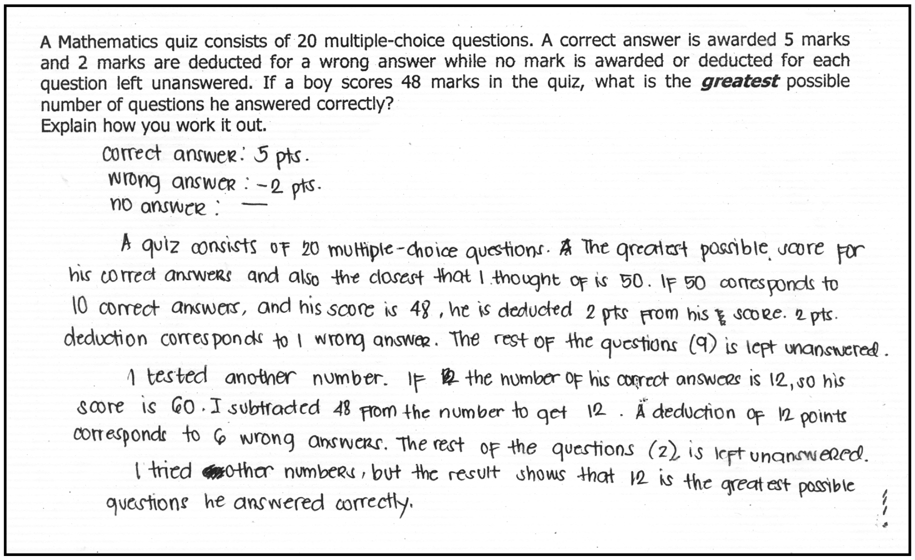
Results and Discussion

Results of this study are presented and discussed in two parts. The first part consists of sample problem solutions actually given by the students that exhibit their different levels of performance in terms of conceptual understanding, procedural knowledge, and problem solving skills and strategies. The second part consists of the evaluation of the levels of performance of the five participants of the study.

Problem Solutions that Exhibit Conceptual Understanding

Proficient. To be proficient in conceptual understanding, the student must have used all relevant information to solve the problem. That is, the student's answer is consistent with the question or problem and the student is able to translate the problem into appropriate mathematical language.

The work of Student D in problem number 2 illustrates this as shown in figure 1. Student D thoroughly investigated the situation and was able to use all applicable information related to problem number 2, like listing the equivalent points for correct, wrong and no answer. Moreover, she tested if the numbers would satisfy the condition. Finally, she came up with the correct greatest possible number of questions the boy had answered correctly in a quiz.


A Mathematics quiz consists of 20 multiple-choice questions. A correct answer is awarded 5 marks and 2 marks are deducted for a wrong answer while no mark is awarded or deducted for each question left unanswered. If a boy scores 48 marks in the quiz, what is the ***greatest*** possible number of questions he answered correctly?
Explain how you work it out.

correct answer: 5 pts.
wrong answer: -2 pts.
no answer: —

A quiz consists of 20 multiple-choice questions. The greatest possible score for his correct answers and also the closest that I thought of is 50. If 50 corresponds to 10 correct answers, and his score is 48, he is deducted 2 pts from his score. 2 pts. deduction corresponds to 1 wrong answer. The rest of the questions (9) is lept unanswered.

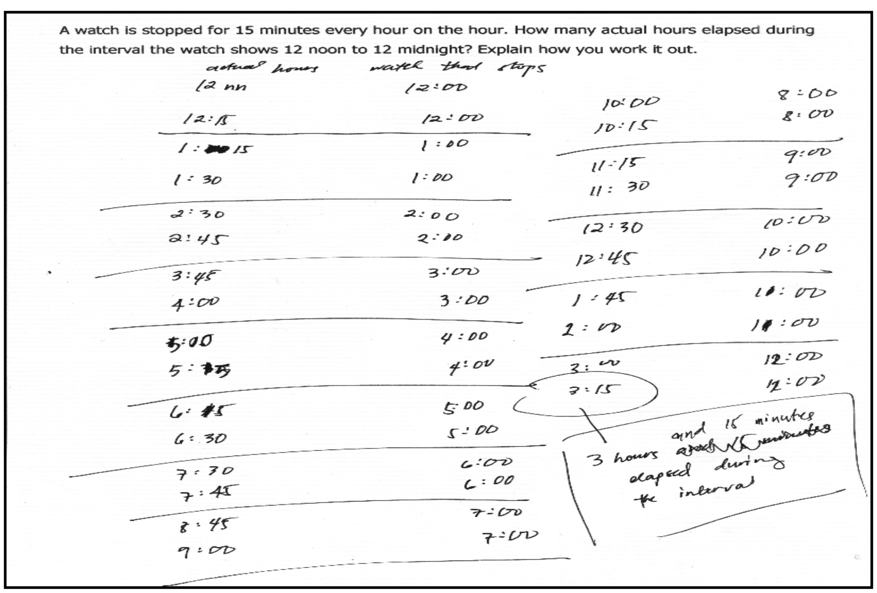
I tested another number. If the number of his correct answers is 12, so his score is 60. I subtracted 48 from the number to get 12. A deduction of 12 points corresponds to 6 wrong answers. The rest of the questions (2) is left unanswered.

I tried other numbers, but the result shows that 12 is the greatest possible questions he answered correctly.

Fig. 1. Student D's solution to problem number 2.

Apprentice. An apprentice in conceptual understanding is one who extracts the essence of the problem, but is unable to use this information to solve the problem. It could be that the student is only partially able to make connections between or among the concepts; the student's solution is not fully related to the question; or the student understands one portion of the task, but not the complete task.

The work of Student E in problem 6 illustrates an apprentice understanding of concepts. Student E seemed to understand one portion of the problem, but not the complete task as revealed in her solution. Student E failed to analyze properly the question: How many actual hours elapsed during the interval the watch shows 12 noon to 12 midnight? She forgot to add 12 and 3 that will result to the correct answer which is 15 hours. She even added another 15 minutes to 3 hours without noticing that she was already on the 12th hour.


A watch is stopped for 15 minutes every hour on the hour. How many actual hours elapsed during the interval the watch shows 12 noon to 12 midnight? Explain how you work it out.

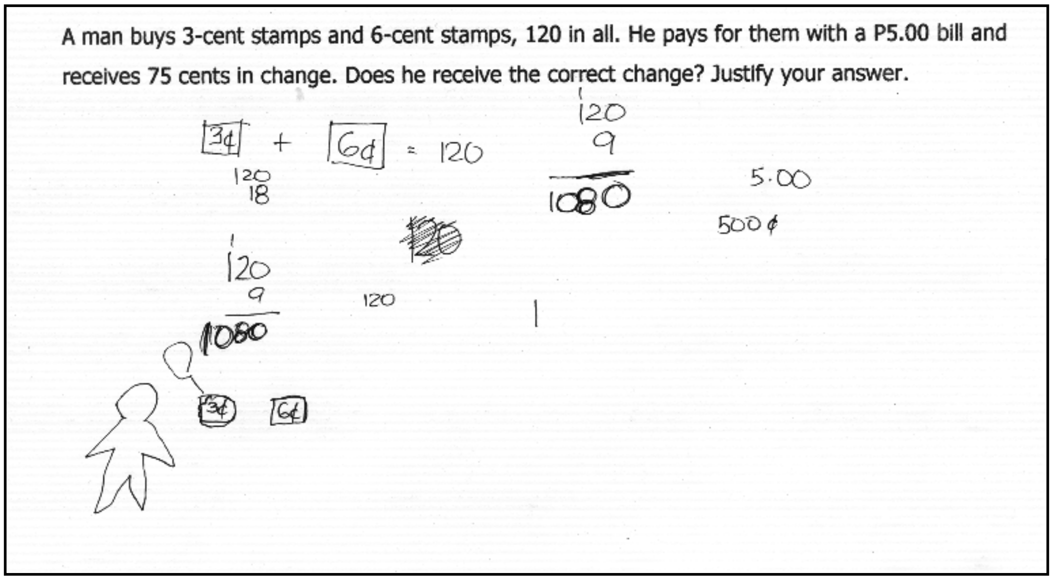
actual hours	watch that stops		
12 nn	12:00	10:00	8:00
12:15	12:00	10:15	8:00
1:15	1:00	11:15	9:00
1:30	1:00	11:30	9:00
2:30	2:00	12:30	10:00
2:45	2:00	12:45	10:00
3:45	3:00	1:45	11:00
4:00	3:00	2:00	11:00
5:00	4:00	3:00	12:00
5:15	4:00	3:15	12:00
6:15	5:00		
6:30	5:00		
7:30	6:00		
7:45	6:00		
8:45	7:00		
9:00	7:00		

3 hours and 15 minutes elapsed during the interval

Fig. 2. Student E's solution to problem number 6.

Novice. If a student's solution is inconsistent or unrelated to the question, then the student is rated as novice in conceptual understanding. This means that the student translates the problem into inappropriate mathematical concepts or the student uses incorrect procedures without understanding the concepts related to the task.

Figure 3 shows how Student B used incorrect procedures. His mathematical statements like 34+64 = 120 show his lack of skill in basic mathematical operations. All other statements were not derived from correct analysis of the given information.



A man buys 3-cent stamps and 6-cent stamps, 120 in all. He pays for them with a P5.00 bill and receives 75 cents in change. Does he receive the correct change? Justify your answer.

$\boxed{3¢} + \boxed{6¢} = 120$

120 × 9 = 1080

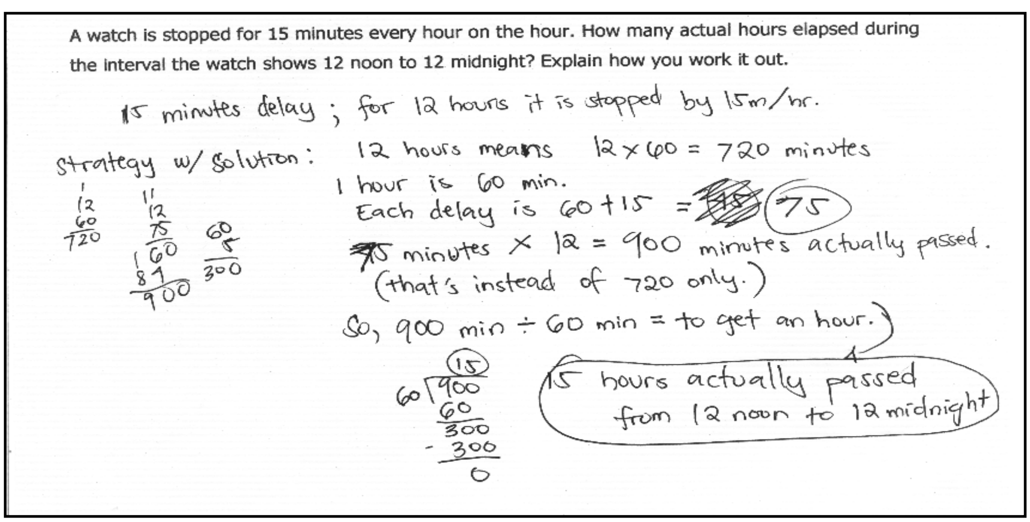
5.00

500¢

Fig. 3. Student B's solution to problem number 7.

Problem Solutions that Exhibit Procedural Knowledge

Proficient. Students who use the mathematical principles and language correctly to solve the problems are classified as proficient in procedural knowledge. Figure 4 illustrates the proficient work of Student A in problem 6. He used appropriate mathematical terms and strategies. He efficiently solved the problem with a good analysis of the given information.



A watch is stopped for 15 minutes every hour on the hour. How many actual hours elapsed during the interval the watch shows 12 noon to 12 midnight? Explain how you work it out.

15 minutes delay ; for 12 hours it is stopped by 15m/hr.

Strategy w/ Solution: 12 hours means 12 × 60 = 720 minutes

1 hour is 60 min.

Each delay is 60 + 15 = 75

75 minutes × 12 = 900 minutes actually passed.

(that's instead of 720 only.)

So, 900 min ÷ 60 min = to get an hour.)

900 ÷ 60 = 15

15 hours actually passed from 12 noon to 12 midnight

Fig. 4. Student A's solution to problem number 6.

Apprentice. A student who is unable to carry out a procedure completely or incorrectly verifies the solution is said to have an apprentice procedural knowledge.

Figure 5 illustrates the work of Student B in problem 6 assessed as apprentice. The inadequacy of Student B's solution is very evident. He failed to realize that the product of 42 and 42 was incorrect as shown in his solution.

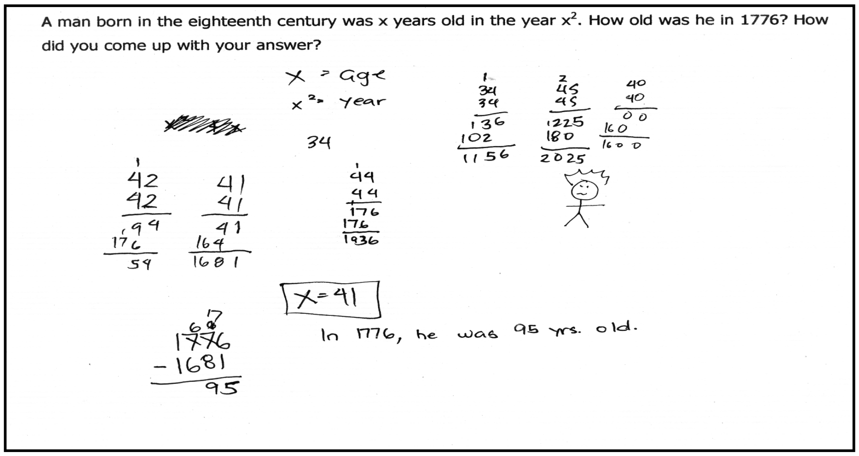


Fig. 5. Student B's solution to problem number 5.

Novice. Students who fail to apply the appropriate procedure, use unsuitable methods or simple manipulation of data in their attempted solution are rated as novice in procedural knowledge. The work of Student A in problem number 1 illustrates this. Student A failed to verify his solution as shown in figure 6. His difference of 150 and 88 is 42 instead of 62. Everything else was wrong after that careless mistake.

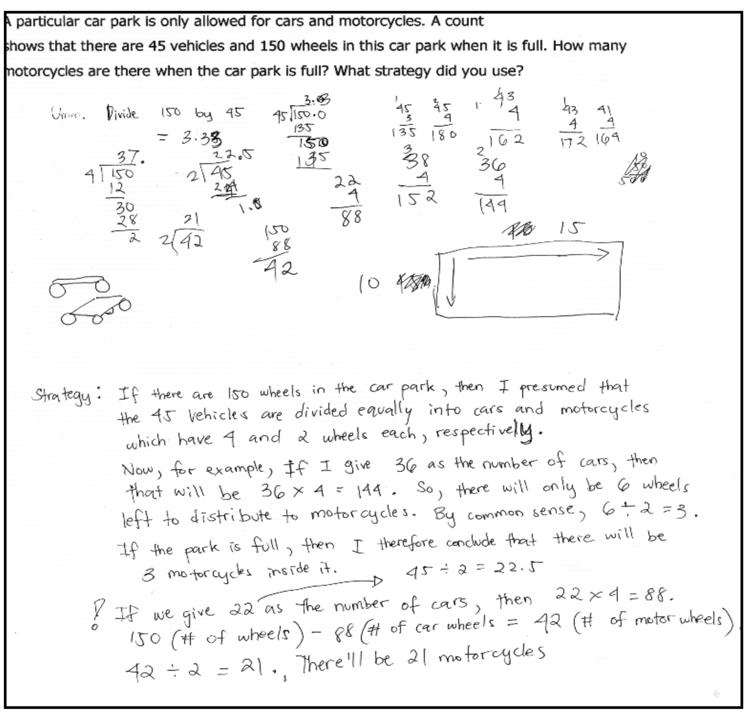


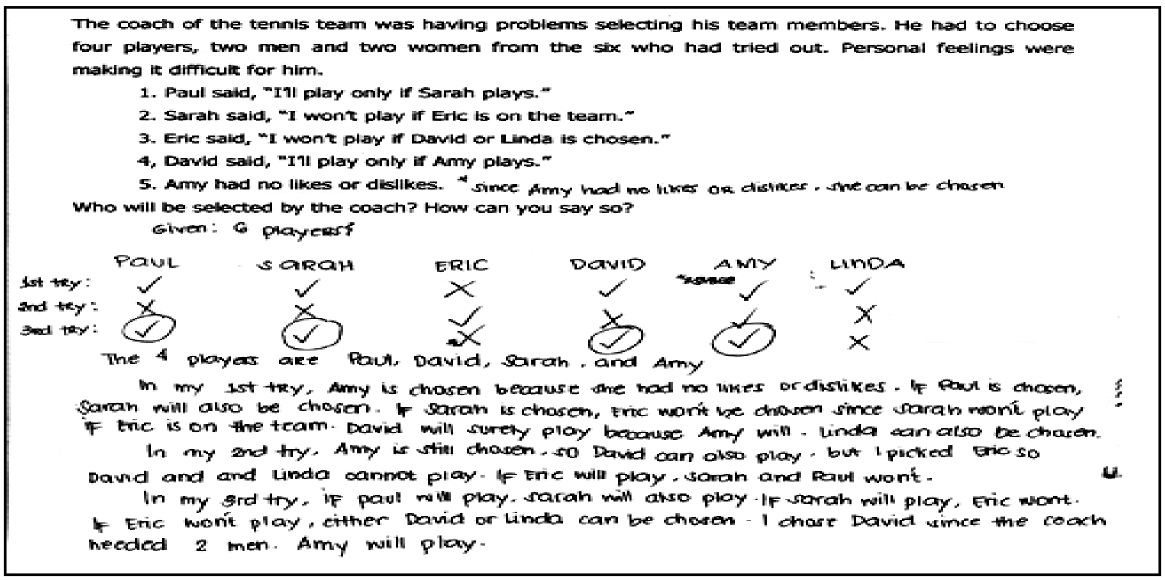
Fig. 6. Student A's solution to problem number 1.

Problem Solutions that Exhibit Problem Solving Strategies

Problem solving requires the use of many skills, often in certain combinations, before the problem is solved. Students demonstrate problem solving strategies with clearly focused solution and good reasoning that lead to a successful resolution of the problem.

Proficient. A proficient rating in problem solving skills and strategies shows evidence of insightful thinking to explore the problem. Some evidences include a clear and focused work, appropriate skills and strategies, extensions and generalizations to the solution of the problem.

The work of Student D in problem number 3 illustrates this. Student D's insightful thinking to explore the problem was evident. Her good reasoning skills were further demonstrated in her solution as shown in figure 7. She has a clear and focused work when she had a table/list of team members and considered the conditions on each of them.


The coach of the tennis team was having problems selecting his team members. He had to choose four players, two men and two women from the six who had tried out. Personal feelings were making it difficult for him.

1. Paul said, "I'll play only if Sarah plays."
2. Sarah said, "I won't play if Eric is on the team."
3. Eric said, "I won't play if David or Linda is chosen."
4, David said, "I'll play only if Amy plays."
5. Amy had no likes or dislikes. * Since Amy had no likes or dislikes, she can be chosen

Who will be selected by the coach? How can you say so?

Given: 6 players

	PAUL	SARAH	ERIC	DAVID	AMY	LINDA
1st try:	✓	✓	✗	✓	✓	✓
2nd try:	✗	✗	✓	✗	✓	✗
3rd try:	✓	✓	✗	✓	✓	✗

The 4 players are Paul, David, Sarah, and Amy

In my 1st try, Amy is chosen because she had no likes or dislikes. If Paul is chosen, Sarah will also be chosen. If Sarah is chosen, Eric won't be chosen since Sarah won't play if Eric is on the team. David will surely play because Amy will. Linda can also be chosen.

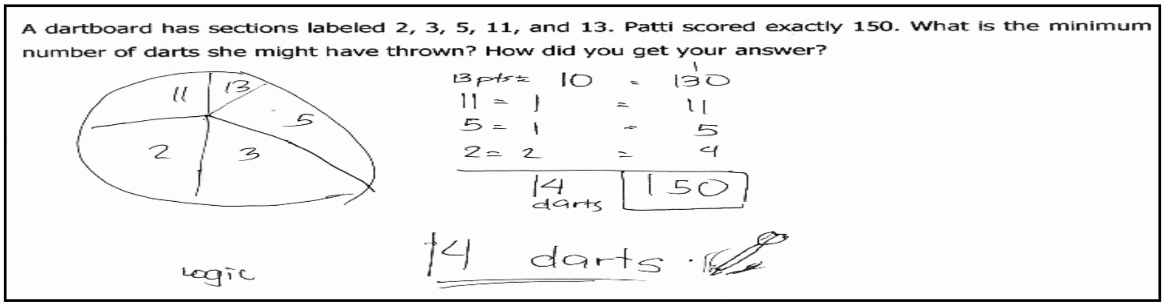
In my 2nd try, Amy is still chosen, so David can also play. but I picked Eric so David and and Linda cannot play. If Eric will play, Sarah and Paul won't.

In my 3rd try, If paul will play, Sarah will also play. If Sarah will play, Eric won't. If Eric won't play, either David or Linda can be chosen. I chose David since the coach needed 2 men. Amy will play.

Fig. 7. Student D's solution to problem number 3.

Apprentice. An apprentice in problem solving skills and strategies is one whose skills and strategies have some focus, but clarity is limited. A student shows routine or partial use of skills and strategies if the student applies a strategy which is only partially useful; the student's strategy is not fully executed; the student starts the problem appropriately, but changes to an incorrect focus; or the student recognizes the pattern or relationship, but proceeds incorrectly.

The work of Student B in problem number 10 illustrates this as shown in figure 8. Student B had some focus on the problem since he knew that the sum must be exactly 150. He had a strategy that was partially useful and possible source of partially correct answer. Student B failed to exhaust the number of 13's in 150.


A dartboard has sections labeled 2, 3, 5, 11, and 13. Patti scored exactly 150. What is the minimum number of darts she might have thrown? How did you get your answer?

11 13 5 2 3

13pts = 10 = 130
11 = 1 = 11
5 = 1 = 5
2 = 2 = 4

14 darts | 150

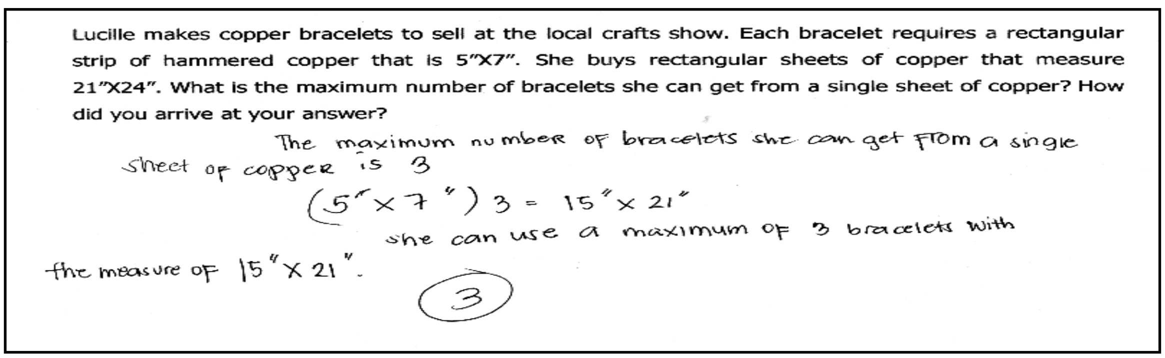
14 darts.

logic

Fig. 8. Student B's solution to problem number 10.

Novice. If the skills and strategies lack a central focus and the details are sketchy and not present, then the student is rated as novice in problem solving strategies. Limited evidence of skills and strategies include unrecorded procedures, random strategies, and failure to see patterns, relationships and alternative solutions.

The work of student D in problem number 9 illustrates this. Student D failed to see alternative solutions that the problem required. She did not fully explore the problem nor look for concepts, patterns or relationships as shown in figure 9.


Lucille makes copper bracelets to sell at the local crafts show. Each bracelet requires a rectangular strip of hammered copper that is 5"X7". She buys rectangular sheets of copper that measure 21"X24". What is the maximum number of bracelets she can get from a single sheet of copper? How did you arrive at your answer?

The maximum number of bracelets she can get from a single sheet of copper is 3

(5" x 7") 3 = 15" x 21"

she can use a maximum of 3 bracelets with the measure of 15" x 21".

3

Fig. 9. Student D's solution to problem number 9.

Components of Problem Solving

The mean scores for each of the components shown in table 4 were used to assess the students' level of performance.

Three of the five students were proficient problem solvers. They were Student A, C and E. They all got an average score of greater than four while Student D was an intermediate problem solver between apprentice and proficient levels who got an average score of 4. Only Student B who got an average score of 3.1 was found out to be an apprentice problem solver.

TABLE 4 Mean Scores obtained by the students in the three components of problem solving

Students	Components	Mean Score on Twelve Problems	Average	Level of Performance
A	Conceptual Understanding	4.5	4.4	Proficient
	Procedural Knowledge	4.4		
	Problem Solving Skills and Strategies	4.2		
B	Conceptual Understanding	3.3	3.1	Apprentice
	Procedural Knowledge	3.2		
	Problem Solving Skills and Strategies	2.8		
C	Conceptual Understanding	4.6	4.6	Proficient
	Procedural Knowledge	4.6		
	Problem Solving Skills and Strategies	4.7		
D	Conceptual Understanding	4.1	4.0	Apprentice/ Proficient
	Procedural Knowledge	4.0		
	Problem Solving Skills and Strategies	4.0		
E	Conceptual Understanding	4.3	4.3	Proficient
	Procedural Knowledge	4.3		
	Problem Solving Skills and Strategies	4.3		

Scale		
	4.2 – 5.0	Proficient
	3.4 – 4.1	Apprentice/Proficient
	2.6 – 3.3	Apprentice
	1.8 – 2.5	Novice/Apprentice
	1.0 – 1.7	Novice

Table 5 shows the different strategies employed by the students in twelve non-routine problems. As can be seen from the table, some of the problems were solved using a combination of two or three strategies. Each student employed at least four problem solving strategies on the twelve non-routine problems.

Seven out of the eight problem solving strategies were used at least once to solve the twelve non-routine problems, "Making a Model or Diagram" being the most frequently used strategy.

Student A utilized "Making a Model or Diagram" (MD) by drawing diagrams or sketching figures in eight problems with a combination of "Make a Table, Chart, or List" (TCL), "Consider a Simple Case" (SC), Eliminate (E), Compute or Simplify (CS), and "Guess, Check and Revise" (GCR).

There were also cases when students used the same strategy on one problem but at different approaches. Student B, Student C, Student D and Student E employed TCL in problem number 6 but as shown in their worksheets, there were slight differences in the way they used the strategy. There were three out of the twelve problems that were approached using three different strategies. These are problems 8, 9, and 10. Only Student B and Student D used different strategies in problem number 3 while the rest used the same strategy which is Consider a Simple Case" (SC).

TABLE 5 Summary of the strategies employed by the students in twelve non-routine problems

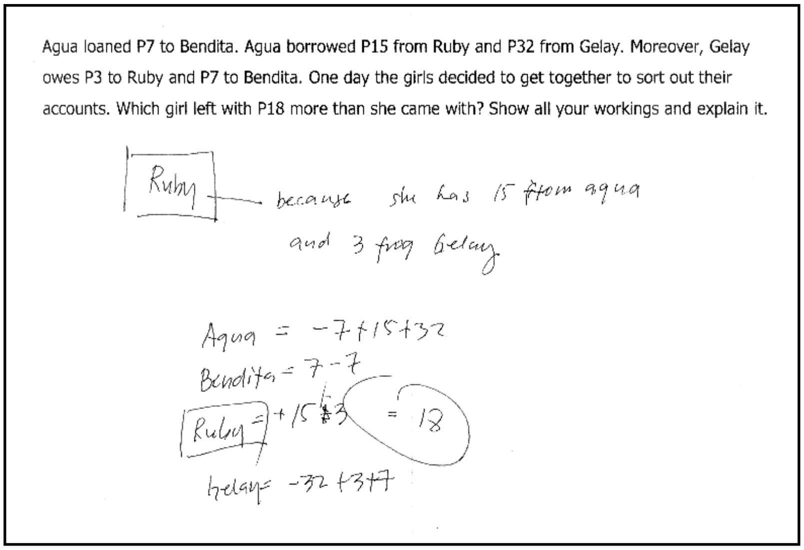
Problem	**Student**				
	A	B	C	D	E
1	GCR/MD	GCR/MD	GCR	GCR	MD
2	GCR/E	GCR	MD/GCR	GCR	GCR
3	SC	MD	SC	TCL/GCR	SC
4	SC/MD	TCL/MD	SC	TCL/MD	CS
5	GCR/CS	GCR/CS	GCR/CS	CS	MD
6	F/SC	TCL	TCL	TCL	TCL
7	MD/GCR	CS	MD	GCR	MD
8	MD/GCR	MD/GCR/TCL	MD/GCR	GCR	GCR
9	MD/CS	MD/TCL/CS	MD/CS	CS	MD/TCL
10	MD/TCL	TCL/MD	MD/CS	SC	MD/TCL/CS
11	GCR/MD	MD/GCR	MD/SC	SC	MD/SC
12	MD/GCR	GCR	MD/GCR	GCR/ TCL	GCR/MD

Legend:

CS – Compute or Simplify
SC – Consider a Simple Case
MD – Make a Model
GCR – Guess, Check, and Revise
F – Use a Formula
TCL – Make a Table, Chart, or List
E – Eliminate
LP – Look for Patterns

Compute or Simplify (CS) – includes straightforward application of arithmetic rules or order of operations

Student E used CS in problem number 4 as shown in her solution ("fig. 10"), she simply added 15 and 3 and obtained 18 by which these amounts are borrowed from Ruby by Agua and Gelay, respectively.



Agua loaned P7 to Bendita. Agua borrowed P15 from Ruby and P32 from Gelay. Moreover, Gelay owes P3 to Ruby and P7 to Bendita. One day the girls decided to get together to sort out their accounts. Which girl left with P18 more than she came with? Show all your workings and explain it.

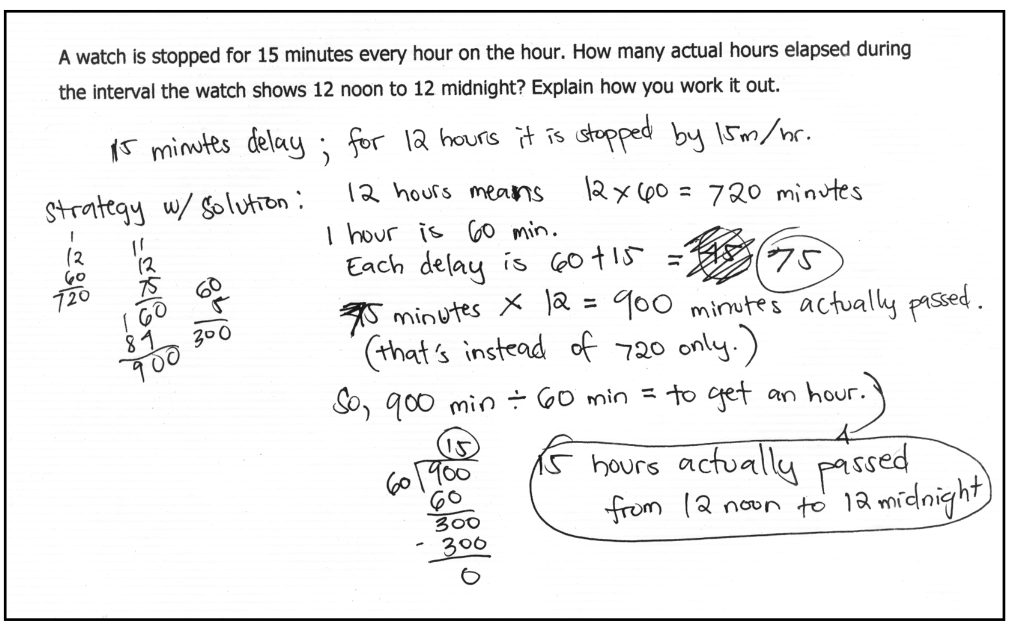
Ruby — because she has 15 from agua and 3 from Gelay

Agua = −7+15+32
Bendita = 7−7
Ruby = +15+3 = 18
Gelay = −32+3+7

Fig. 10. Student E's solution to problem number 4.

Use a Formula (F) – involves substituting values into a formula or selecting the proper formula to use.

Among the five students, only Student A used F. He employed F in problem number 6 as shown in his solution ("fig. 11"). He converted hours to minutes using the conversion 1hr = 60 min.



A watch is stopped for 15 minutes every hour on the hour. How many actual hours elapsed during the interval the watch shows 12 noon to 12 midnight? Explain how you work it out.

15 minutes delay ; for 12 hours it is stopped by 15m/hr.

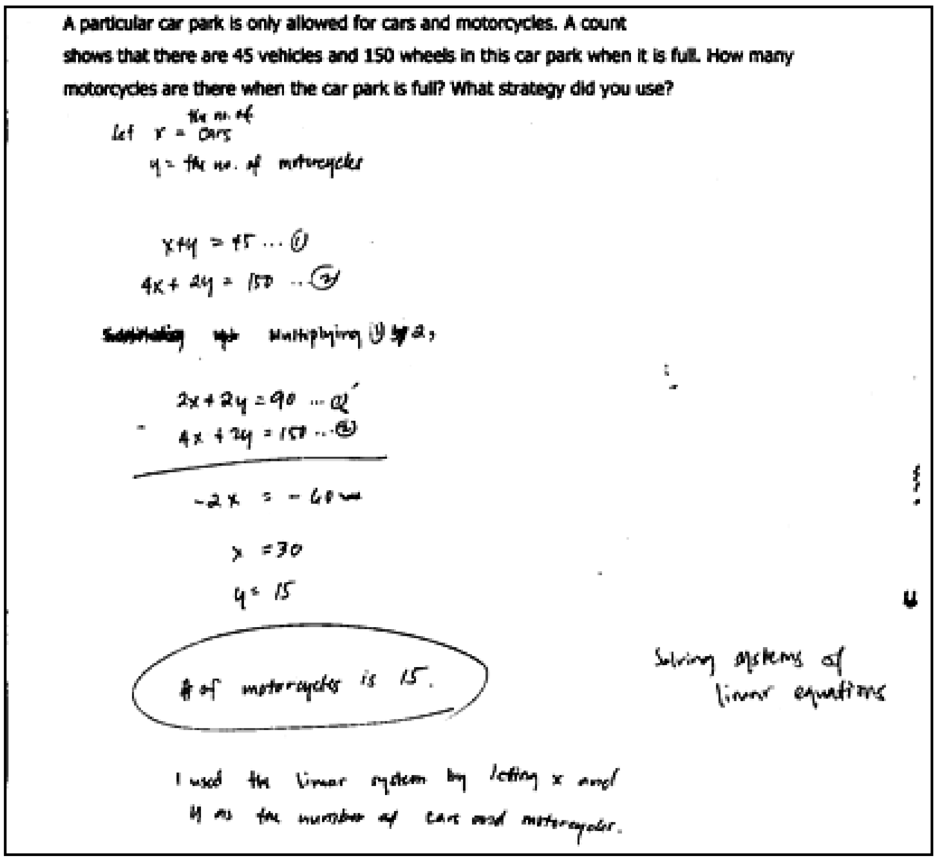
Strategy w/ solution: 12 hours means 12 × 60 = 720 minutes
1 hour is 60 min.
Each delay is 60 + 15 = 75
75 minutes × 12 = 900 minutes actually passed.
(that's instead of 720 only.)
So, 900 min ÷ 60 min = to get an hour.

15 hours actually passed from 12 noon to 12 midnight

Fig. 11. Student A's solution to problem number 6.

Make a Model or Diagram (MD) – includes use of objects, drawings or sketches, acting out, writing an equation.

Student E used MD seven times. In problem number 1, she wrote an equation representing the number of vehicles and wheels. She also used the variable x and y to represent the number of cars and motorcycles respectively. As seen in her solution ("fig. 12"), she was able to set up a system of linear equations in two variables in order to get the number of motorcycles. She used algebraic skills to solve the system.


A particular car park is only allowed for cars and motorcycles. A count shows that there are 45 vehicles and 150 wheels in this car park when it is full. How many motorcycles are there when the car park is full? What strategy did you use?

Let x = the no. of cars
y = the no. of motorcycles

$x+y = 45 \ldots (1)$
$4x + 2y = 150 \ldots (2)$

~~Substitution~~ by multiplying (1) by 2,

$2x+2y = 90 \ldots (1)'$
$4x + 2y = 150 \ldots (2)$
$-2x = -60$
$x = 30$
$y = 15$

# of motorcycles is 15.

Solving systems of linear equations

I used the linear system by letting x and y as the number of cars and motorcycles.

Fig. 12. Student E's solution to problem number 1.

She used MD together with "Make a Table, Chart, or List" (TCL), and Compute or Simplify (CS) strategy to solve problem number 9. She first drew a table representing rectangular strip of copper. She divided the length with the given measure of 7 and the width by 5 ("fig. 13"). She used her drawing and straightforward application of arithmetic rules to figure out the number of bracelets she can get from a single sheet of copper.

She used MD together with "Make a Table, Chart, or List" (TCL), and Compute or Simplify (CS) strategy to solve problem number 9. She first drew a table representing rectangular strip of copper. She divided the length with the given measure of 7 and the width by 5 ("fig. 13"). She used her drawing and straightforward application of arithmetic rules to figure out the number of bracelets she can get from a single sheet of copper.

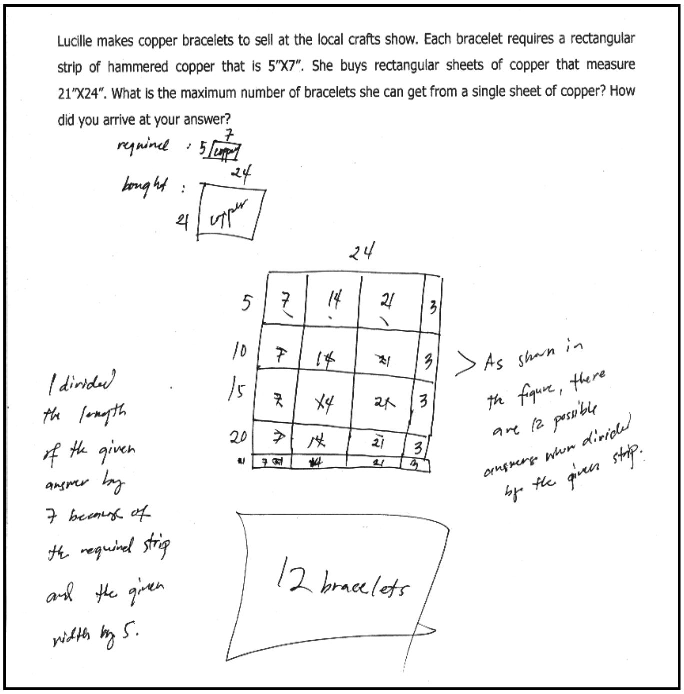


Fig. 13. Student E's solution to problem number 9.

In problem number 10, in order to visualize the problem, she drew a figure which represented a dartboard and numbered them 2, 3, 5, 11, and 13 ("fig. 14").

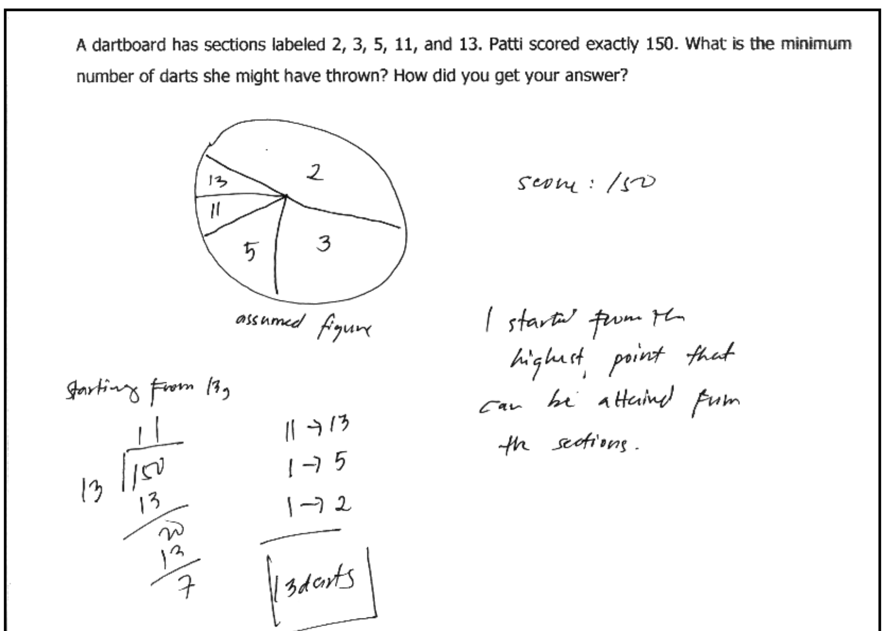


Fig. 14. Student E's solution to problem number 10.

Like what she did in problem number 1, she wrote an equation and computed for the answer backwards in problem number 11 (“fig. 15”).

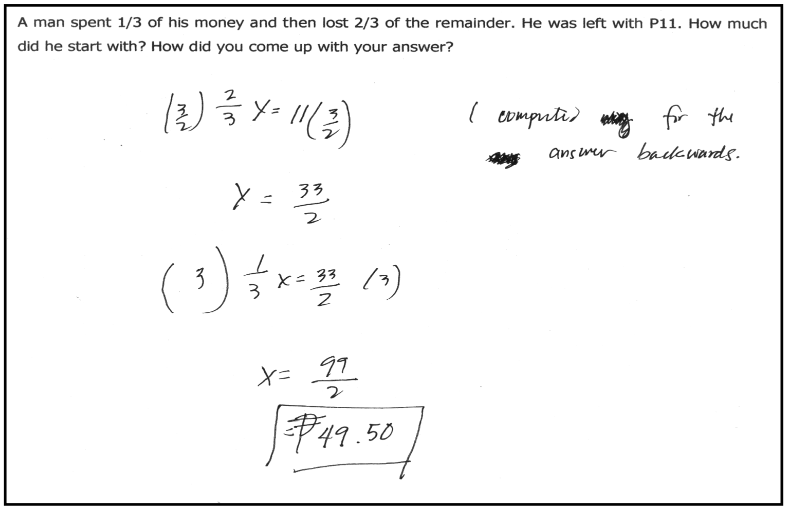


Fig. 15. Student E’s solution to problem number 11.

Both Student C and Student E utilized MD on exactly the same problem, problem number 7. They both used system of linear equations in two variables to solve the problem. Student C employed substitution method while Student E used elimination method to solve the system (“fig. 16”).

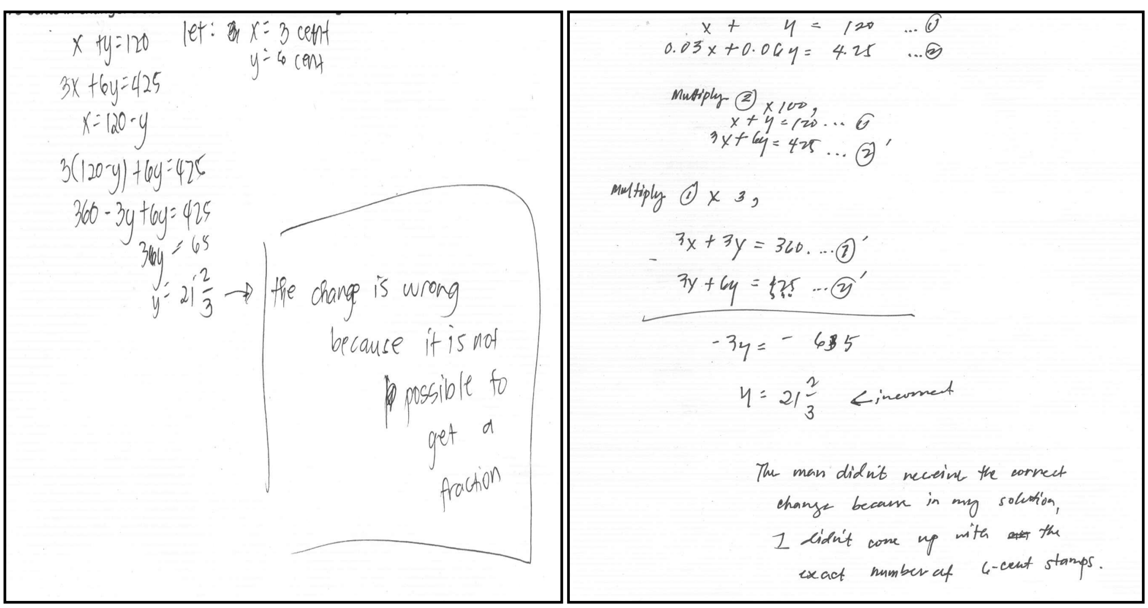


Fig. 16. Student C’s and student E’s solutions to problem number 7.

Make a Table, Chart, or List (TCL) – organizing the data by making a table, chart, graph, or list.

In problem number 6 four students utilized TCL to solve the problem ("fig. 17-20"). All were able to construct a table to organize their solution. As shown on their workings, they were able to make two columns that show their interpretation of the problem.

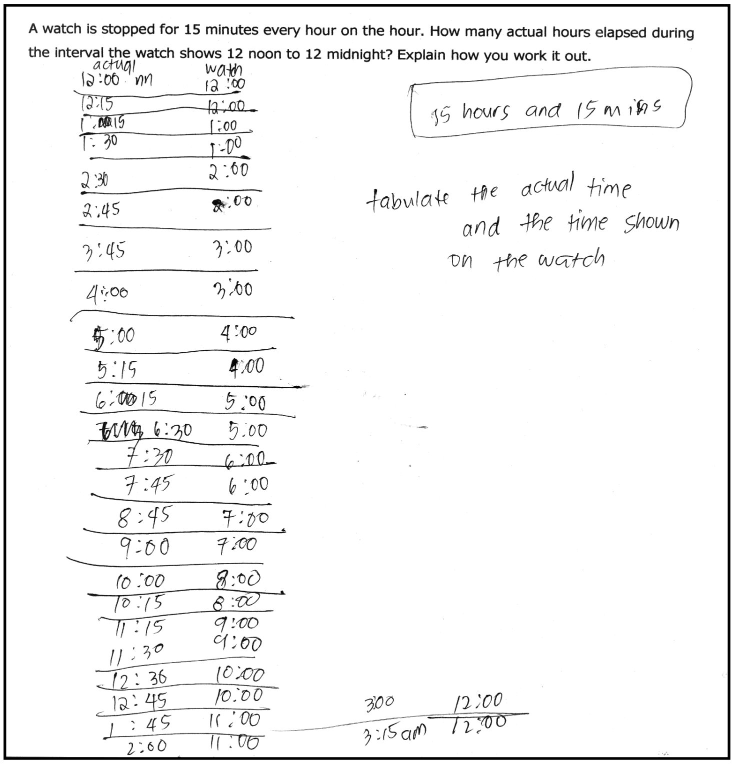


Fig. 17. Student C's solution to problem number 6.

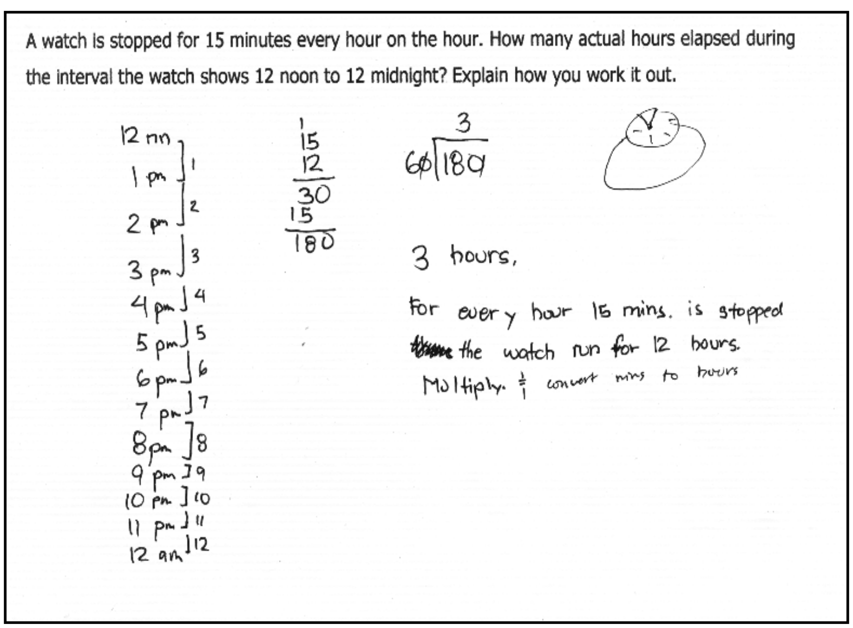


Fig. 18. Student B's solution to problem number 6.

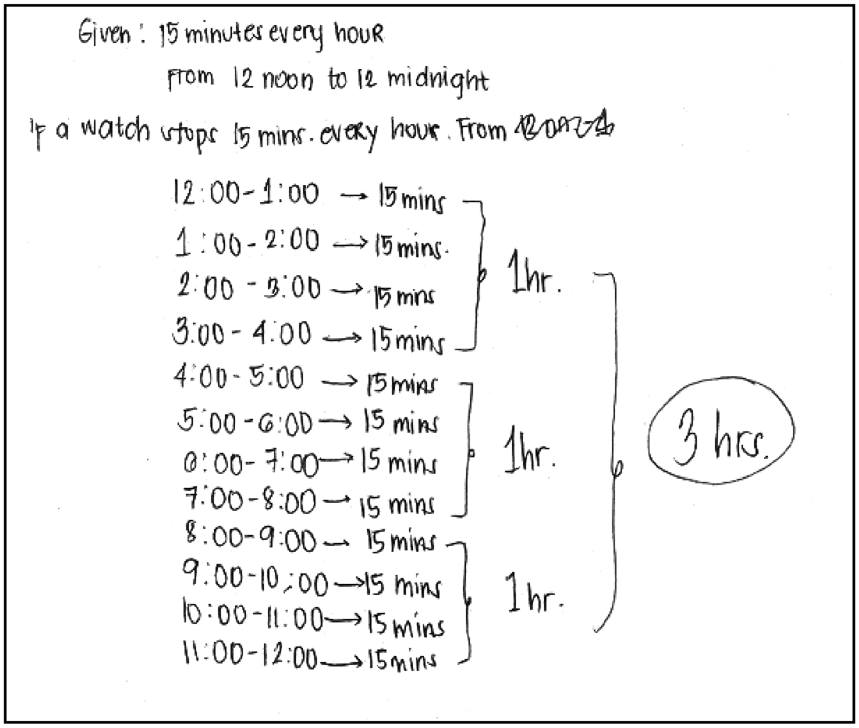


Fig. 19. Student D's solution to problem number 6.

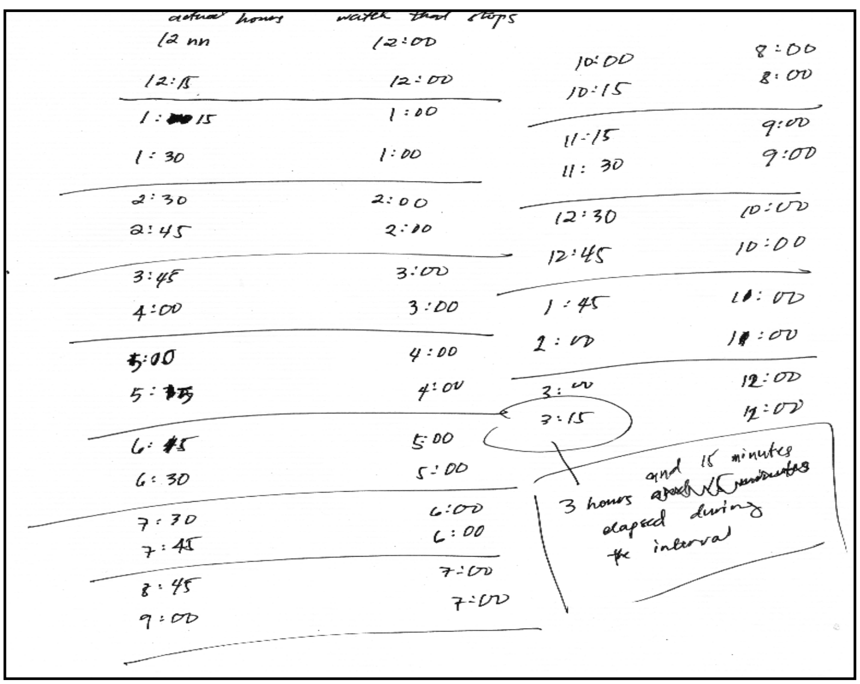


Fig. 20. Student E's solution to problem number 6.

Only Student A did not use TCL. Instead, he used a Formula (F) and a Simpler Case (SC) strategy to get the actual number of hours ("fig. 21").

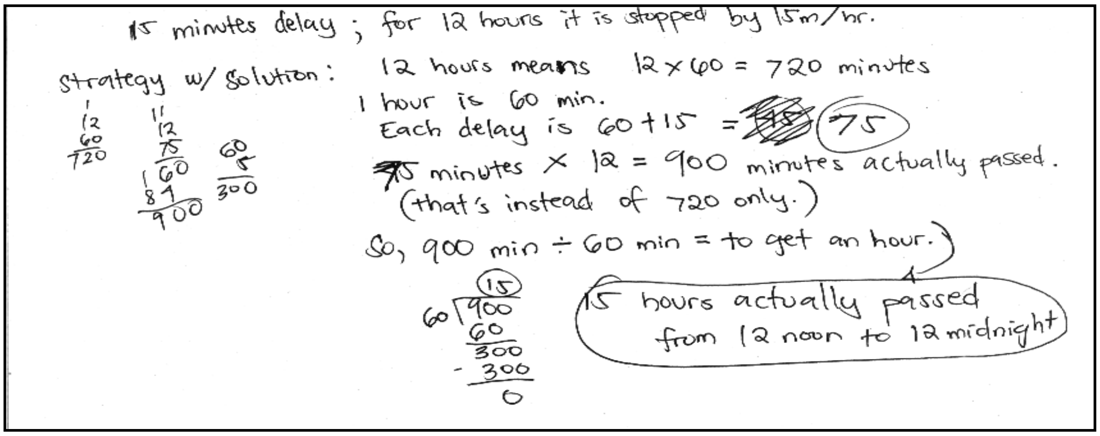


Fig. 21. Student A's solution to problem number 6.

Guess, Check and Revise (GCR) – making a reasonable guess, checking the guess, and revising the guess if necessary.

All the students utilized GCR in most of the problems. Student D used GCR in problem numbers 1, 2, 7, and 8. It was very evident that she did not use any strategy except GCR as seen in her worksheets ("fig. 22-25").

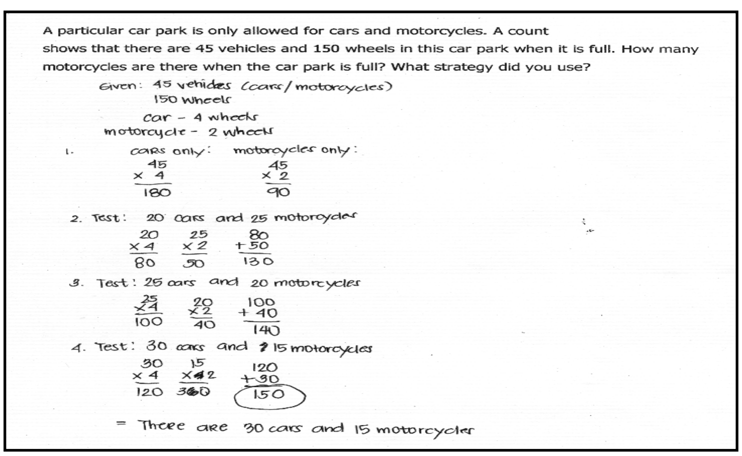


Fig. 22. Student D's solution to problem number 1.

Student D tried some numbers and checked whether they satisfy the condition on the problem.

A Mathematics quiz consists of 20 multiple-choice questions. A correct answer is awarded 5 marks and 2 marks are deducted for a wrong answer while no mark is awarded or deducted for each question left unanswered. If a boy scores 48 marks in the quiz, what is the ***greatest*** possible number of questions he answered correctly?
Explain how you work it out.

correct answer: 5 pts.
wrong answer: −2 pts.
no answer: —

A quiz consists of 20 multiple-choice questions. ~~A~~ The greatest possible score for his correct answers and also the closest that I thought of is 50. If 50 corresponds to 10 correct answers, and his score is 48, he is deducted 2 pts from his ~~s~~ score. 2 pts. deduction corresponds to 1 wrong answer. The rest of the questions (9) is left unanswered.

I tested another number. If ~~12~~ the number of his correct answers is 12, so his score is 60. I subtracted 48 from the number to get 12. A deduction of 12 points corresponds to 6 wrong answers. The rest of the questions (2) is left unanswered.

I tried another numbers, but the result shows that 12 is the greatest possible questions he answered correctly.

Fig. 23. Student D's solution to problem number 2.

Given: 3-cent stamps
6-cent stamps > 120
₱5.00 – paying amount
75 ¢ – change

1st: 60 3-cent   60 × 3 cents = 180 × 100 = ₱1.80
60 6-cent   60 × 6 cents = 360 × 100 = ₱3.60
₱5.40

₱5.40 exceeds the amount

2nd: 40 6-cent   40 × 6 = 240 × 100 = ₱2.40
80 3-cent   80 × 3 = 240 × 100 = ₱2.40
₱4.80

We are aiming for ₱4.25 to check if he really receives 75 cents

3rd: 20 6-cent   20 × 6 = 120 × 100 = ₱1.20
100 3-cent   100 × 3 = 300 × 100 = ₱3.00
₱4.20 closest amount to ₱4.25

He didn't receive the correct change. His change should be 80 cents

Fig. 24. Student D's solution to problem number 7.

Given: ₱200
at least 5 hotdogs (n > 5) ₱20@
at least 5 softdrinks (n > 5) ₱10@

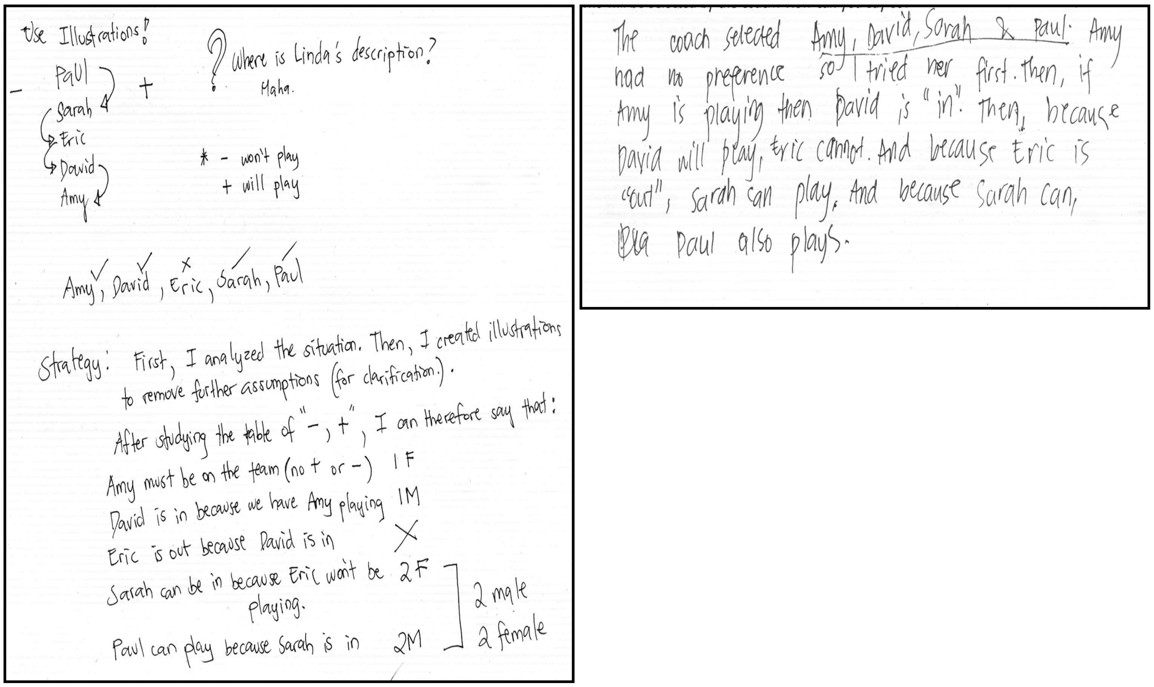
1st: 6 hotdogs × 20 = ₱120   200 − 120 = 80
8 softdrinks × 10 = ₱80
₱200

2nd: 7 hotdogs × 20 = ₱140
6 softdrinks × 10 = ₱60
₱200

Fig. 25. Student D's solution to problem number 8.

Consider a Simpler Case (SC) – includes rewording the problem, using smaller numbers, using a more familiar problem setting, dividing the problem into simpler problems, or working backwards.

Student C and A were able to use effectively SC in solving problem number 3. Their solutions somehow made the problem appear simpler to them than it looked like ("fig. 26"). They divided the problem into simpler statements.


Use Illustrations!
Paul
Sarah
Eric
David
Amy
+
Where is Linda's description? Haha.
* - won't play
+ will play
Amy, David, Eric, Sarah, Paul
Strategy: First, I analyzed the situation. Then, I created illustrations to remove further assumptions (for clarification).
After studying the table of "–, +", I can therefore say that:
Amy must be on the team (no + or –) 1F
David is in because we have Amy playing 1M
Eric is out because David is in
Sarah can be in because Eric won't be playing. 2F
Paul can play because Sarah is in 2M
2 male
2 female

The coach selected Amy, David, Sarah & Paul. Amy had no preference so I tried her first. then, if Amy is playing then David is "in". Then, because David will play, Eric cannot. And because Eric is "out", Sarah can play. And because Sarah can, Paul also plays.

Fig. 26. Student C's an A's solutions to problem number 3.

Looking for Patterns (LP) - involves determining certain common characteristics that can be generalized and used to solve the problem.

No one used this strategy in solving any of the 12 non-routine problems.

Table 6 shows the frequency of problem solving strategies employed by the students belonging to the same level of performance. As gleaned on the table, "Make a Model or Diagram" was the most frequently used strategy by both proficient and apprentice problem solvers, while the "Guess, Check and Revise" was used frequently by the transitional problem solver between apprentice and proficient levels. Only Student A used "Eliminate" and "Use a Formula" among the five students in solving the twelve non-routine problems. The proficient problem solvers employ at least 5 different problem solving strategies.

TABLE 6 Frequency of Problem Solving Strategies of Students with the same Level of Performance

Level of Performance	Student	Strategies						
		GCR	MD	E	SC	CS	F	TCL
Proficient	A	7	8	1	3	2	1	1
	C	5	7		3	3		1
	E	3	7		2	2		3
Apprentice/ Proficient	D	6	1		2	2		4
Apprentice	B	6	7			3		5

Conclusions and Recommendations

Each student in this study employed singly or in certain combinations at least four problem-solving strategies on the twelve non-routine problems. Seven out of the eight possible problem solving strategies were used at least once to solve the twelve non-routine problems, "Making a Model or Diagram" being the most frequently used strategy. Some of the problems were solved using a combination of 2 or 3 strategies like Making a Model or Diagram, Making a Table, Chart, or List and Computing or Simplifying. Based on the solutions of the students, a problem solving strategy could be employed by a student in different ways depending on the complexity of the problem. There were also cases when students used the same strategy on one problem but with different approaches. There were three out of the twelve problems that were approached using three different strategies. Despite different approaches, all five students successfully solved problem numbers 8 and 12. This indicates that the participants of this study know that a single problem can be solved in more than one way. After evaluating the students' problem solving abilities, this study shows that three of the five students were proficient problem solvers while one of them was a transitional problem solver between apprentice and proficient levels. One student was an apprentice problem solver. "Making a Model or Diagram" was the most frequently used strategy by both proficient and apprentice problem solvers.

This study showed that junior high school students could employ different problem solving strategies without prior instruction, if given the chance to solve non-routine problems. All five students had a high level of problem solving ability, thus attaining a high level of performance in the problem solving tasks. These high performing students became enthusiastic in solving the problems when they were allowed to use any strategy in finding the answer.

Since the students who participated in this study all belong to high-ability group, it is suggested that a similar study be done with students belonging to low-ability group. More supporting evidence is needed to show that the problem-based instruction and learning is more effective in nurturing the critical and analytical thinking skills of the students in any ability level. Exposing the students to non-routine problems, can develop students' mathematical reasoning power and foster their understanding that mathematics is a creative endeavor. Also non-routine problems provide students a realistic situation where they are challenged to use higher order thinking skills including their critical and creative thinking.

Academic institutions may consider immersing students in a setting where they can find situations of interest to them. Schools should start introducing the students to problems that lend themselves to long-term, thorough analysis at different levels of intellectual accomplishments. Problem solving activity should be embedded in all aspects of learning situations. In doing so, it is important to document the results of implementing such changes in order to inform school administrators and teachers about the mathematical thinking of Filipino students and in order to help them make curricular decisions.

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