ELASTIC-PLASTIC ANALYSIS OF UNDERGROUND OPENINGS BY FINITE ELEMENT METHOD

Foreword

by Mark Albert H. Zarco
Institute of Civil Engineering
University of the Philippines Diliman

As an engineer and an educator, Salvador F. Reyes has made an indelible mark in Philippine Engineering. He started teaching in 1952 immediately after completing his Bachelor of Science in Civil Engineering. He earned a Certificate in Highway Traffic from Yale University, and Master of Science in Civil Engineering and Doctor of Philosophy in Civil Engineering from the University of Illinois Urbana-Champaign. He then returned to UP to continue his teaching career until his retirement in 1995. During this period, he assumed the post Director of Graduate Studies of the UP College of Engineering, and was instrumental in setting up the Transport Training Center. Even after his retirement, he continued to lecture on CE courses until 2000, and is still actively involved in geotechnical consultation work up to the present. Reyes had an illustrious career as a practitioner in various fields which include being a structural engineer/consultant to the Mount Samat Shrine's Dambana ng Kagitingan, Hotel Intercontinental, and Metrobank Building; being dam designer and materials specialist for the Magat Dam, Pantabangan Dam, and Pagudpud Impounding Basins; consultant for slope stability and related problems of the Ambuklao Dam Spillway, Caliraya Dam, Semirara Coal Pit, Tagaytay Highlands and Vista de Loro Heights; and geotechnical and foundations engineering consultant of the Mandaue Opon Bridge, San Juanico Bridge, Marinduque Nickel Project, MWSS Water Supply Project, and Subic Naval Base, to name few. For his achievement, Dr Reyes received the UP Professional Award in Engineering and the Outstanding Civil Engineer Award from the Professional Regulatory Commission.

The following article entitled "Elastic-plastic analysis of underground openings by finite element method" appeared in the proceedings of the 1st Congress of the International Society of Rock Mechanics held on September 1966 in Lisbon Portugal. The paper was based on Dr. Reyes' doctoral dissertation, which was conducted under the supervision of Prof. Donald U. Deere. This paper was among the very first applications of the finite element method in area of rock mechanics and geotechnical engineering, and became the motivation for application of the finite element method to the solution of elasto-plastic problems. The paper is considered a seminal work, and has been widely cited in numerous authors, most notably Zienkiewicz, Valliapan and King (1969), and Desai and Christian (1977).

Elastic-plastic analysis of underground openings by the finite element method

Analyse élasto-plastique des cavités souterraines par la méthode des éléments finis

Elastische-plastische Analyse von unterirdischen Öffnungen durch die Methode der begrenzten Elementen

by Salvador F. Reyes,
Associate Professor of Civil Engineering, University of the Philippines
and Don U. Deere,
Professor of Civil Engineering and of Geology, University of Illinois

Summary

A method of analyzing stress concentrations and displacements around cavities of arbitrary shape was developed, wherein the plasticity of the material was assumed to be governed by parameters analogous to angle of internal friction and cohesion. The yield function plots as a cone in principal stress space; the stress-strain relations were obtained by regarding the function as a plastic potential. The displacement method of finite element analysis (with triangular elements and linear displacement functions) was employed in a step by step application of load. At each step, increments of displacement which satisfy equilibrium were determined by a recursive process. Stress concentrations around circular openings in an infinite medium were analyzed for arbitrary values of yield parameters and initial state of stress. Results appear to be fairly realistic and reflect the effects of volumetric expansion accompanying yielding, as implied by the stress-strain relations used.

Résumé

Les auteurs développent une méthode d'analyse de la concentration des charges et déplacements au voisinage de cavités de formes arbitraires, pour lesquelles ils admettent que la plasticité du materiau est gouvernée par des paramètres de même nature que l'angle de frottement interne et la cohésion. La fonction de passage à l'état plastique a la forme d'un cône dans l'espace des contraintes principales; les relations contraintes-déformations ont été obtenues en considérant la fonction comme un potentiel plastique. La méthode de déplacement de l'analyse d'éléments finis (éléments triangulaires et fonctions de déplacement linéaires) a été utilisée avec application discontinue et progressive des charges. A chaque palier, l'accroissement de déplacement satisfaisant à un nouvel état d'équilibre a été déterminé par récurrence. Les concentrations de contraintes autour d'ouvertures circulaires dans un milieu infini ont été étudiées pour des valeurs arbitraires des paramètres de plasticité et des conditions initiales de contraintes. Les résultats apparaissent bien conformes à la réalité des phénomènes et rendent compte des effets d'expansion volumétrique accompagnant la déformation plastique, conformément aux relations contraintes-déformations utilisées.

Zusammenfassung

Die Verfasser berichten über eine von ihnen entwickelte Methode für die Analyse von Druckkonzentrationen und Druckverschiebungen um Hohlungen von willkürlicher Form herum, wobei sie annehmen, dass die Stoffplastizität von Parametern beherrscht wird, welche dem Winkel von Reibung und Kohäsion analog sind. Die Ergebniss-Funktion zeichnet sich aus als ein Kegel im Hauptdruckraum; die Verhältnisse zwischen Druck und Spannung wurden erhalten, indem man die Funktion als eine plastische Leistungsfähigkeit betrachtet. Die Methode der Verschiebungen-Analyse durch begrenzte Elemente (mit dreieckigen Elementen und Funktionen von linearen Verschiebungen) wurde angewandt durch schrittweisen Zusatz von Ladung. Der jeweilige Zunahme von Verschiebungen nach eingetretenem Gleichgewicht wurde durch einen rücklaeufigen Prozess bestimmt und festgesetzt. Die Druckkonzentrationen um kreisfoermige Öffnungen herum in unbegrenzten Milieu wurden analysiert fuer willkürliche Werte von Stoffplastizität-Parametern und Anfangsstadium von Druck. Die Ergebnisse scheinen der Wirklichkeit ziemlich zu entsprechen, und geben die volumetrische Ausdehnung wieder, welche das Ergebniss begleitet, so wie es durch die angewandten Druck- und Spannungsverhältnisse angedeutet wird.

Introduction

One of the common problems in rock mechanics is that of estimating stress concentrations and displacements that occur around cavities formed in a rock mass. Observations in the field indicate that the use of classical solutions from the theory of elasticity can be quite unrealistic, particularly if the rock is weak or extensively jointed. On the other hand, reduced tangential stresses near the walls of the cavity suggest that an elasto-plastic analysis may provide solutions which are in better agreement with actual field measurements.

Because of the very considerable amount of computations involved in the numerical solution of the problems, not many published results are available at the present time. The classical papers of Allen and Southwell [1] and of Jacobs [9], in which tension plates with holes and notched plates are considered, represent pioneering efforts on the numerical solution of elasto-plastic problems in accordance with the incremental theory of plasticity. Ang and Harper

[2] considered similar problems with the help of a physical model of the governing finite difference equations. More recently, POPE [10] used the finite element method to solve for stresses in tension plates subjected to edge loads. Other examples of the use of the finite element method appear to be based on the deformation theory of plasticity [14]. In all of the foregoing investigations, the Mises criterion of yielding [8] was used.

Because the use of large electronic computers is now becoming quite common, routine elasto-plastic analysis of practical problems in rock mechanics may soon be feasible. For such a general usage, the numerical procedure should have the capability of handling boundary conditions of practical interest. Also, the yield criterion used must account for internal friction in the rock.

The object of the present study was to develop a numerical method for the analysis of two-dimensional (plane strain) problems concerning stress concentrations and displacements in the rock mass around an underground opening. By adopting the finite element method to a step-by-step analysis [3] and employing the generalized Mises criterion [6] which accounts for both internal friction and cohesion, the method can be used for openings of arbitrary shape with any selection of elastic constant, yield parameters, and initial state of stress. Typical solutions were obtained for the purpose of assessing the possible practical value of the elasto-plastic idealization as applied to rocks.

Yield criterion and stress-strain relations

The yield condition used is the following generalization of the Mises criterion for a perfectly plastic material [6]:

$$f = \alpha J_1 + J_2^{1/2} = k \tag{1}$$

where J_1 and J_2 are, respectively, the first invariant of the stress tensor and the second invariant of the stress deviator tensor. The yield parameters, α and k, may be related to the Mohr-Coulomb parameters, cohesion (c) and angle of internal friction (ϕ). Since Eq. 1 plots as a right circular cone in principal stress space, whereas the Mohr-Coulomb plots as a pyramid, such relations must pertain to specific stress states. For example the following have been developed for plane strain [6]:

$$\alpha = \frac{\tan\phi}{(9 + 12\tan\phi)^{1/2}} \qquad k = \frac{3\,c}{(9 + 12\tan\phi)^{1/2}} \tag{2}$$

In the elastic range ($f < k$ or $f = k$ and $\dot f < 0$ in Eq. 1), Hooke's law applies. Expressed in terms of the stress rate tensor, $\dot\sigma_{ij}$ and the elastic strain rate tensor, $\overset{(e)}{\dot\varepsilon}_{ij}$, it may be written as [12]:

$$\overset{(e)}{\dot\varepsilon}_{ij} = \frac{1+\nu}{E}\,\dot\sigma_{ij} - \frac{\nu}{E}\,\dot\sigma_{kk}\,\delta_{ij} \qquad (i, j = 1, 2, 3) \tag{3}$$

where ν and E are, respectively, Poisson's ratio and modulus of elasticity; δ_{ij} is Kronecker's delta; $\dot\sigma_{kk} = \dot\sigma_{11} + \dot\sigma_{22} + \dot\sigma_{33} = \dot\sigma_x + \dot\sigma_y + \dot\sigma_z$.

In the plastic range ($f = k$ and $\dot f = 0$), perfect plasticity is assumed and the plastic strain rate is obtained from the yield criterion as follows:

$$\overset{(p)}{\dot\varepsilon}_{ij} = \dot\lambda\,\frac{\partial f}{\partial\sigma_{ij}} = \dot\lambda\left[\alpha\,\delta_{ij} + \frac{s_{ij}}{2\,J_2^{1/2}}\right]$$

where $\dot\lambda$ is a positive scalar function of the stress and stress rates; s_{ij} is the stress deviator tensor; σ_{ij} is the total stress tensor. Then, the total strain rate is

$$\dot\varepsilon_{ij} = \overset{(e)}{\dot\varepsilon}_{ij} + \overset{(p)}{\dot\varepsilon}_{ij} \tag{4}$$

By combining Eqs. 1, 3 and 4 to eliminate the scalar parameter $\dot\lambda$, the stress-strain relations in the plastic range can be expressed in the form [11]

$$\frac{\dot\sigma_{ij}}{2\,G} = \dot\varepsilon_{ij} - \dot a\,\delta_{ij} - \dot b\,\sigma_{ij} \tag{5}$$

where $\dot a$ and $\dot b$ are the following functions of the current stresses and the material constants:

$$\dot a = \left(\dot\varepsilon_{kk} - \frac{3\alpha\dot W}{K}\right)\left[\left\{\frac{\frac{3K\alpha}{G} - \frac{J_1}{3J_2^{1/2}}}{1 + 9\alpha^2\frac{K}{G}}\right\}\left\{\alpha - \frac{J_1}{6J_2^{1/2}}\right\} - \frac{3\nu K}{E}\,\frac{K}{J_2^{1/2}\left(1 + 9\alpha^2\frac{K}{G}\right)}\right] + \frac{\dot W}{k}\left[\alpha - \frac{J_1}{6J_2^{1/2}}\right]$$

$$\dot b = \frac{1}{2J_2^{1/2}}\left[\frac{\dot W}{k} + \frac{\frac{3K\alpha}{G} - \frac{J_1}{3J_2^{1/2}}}{1 + 9\alpha^2\frac{K}{G}}\left\{\dot\varepsilon_{kk} - 3\alpha\frac{\dot W}{k}\right\}\right]$$

with

$$\dot W = \sigma_{ij}\,\dot\varepsilon_{ij} \quad \text{and} \quad K = \frac{E}{3(1-2\nu)}.$$

For plane strain, it is convenient to express the stress-strain relations in matrix form

$$\dot\sigma = D\,\dot\varepsilon \tag{6}$$

$$\text{where } \dot\sigma^t = [\dot\sigma_x, \dot\sigma_y, \dot\tau], \quad \dot\varepsilon^t = [\dot\varepsilon_x, \dot\varepsilon_y, \dot\gamma].$$

D is the 3×3 symmetric stress-strain matrix.

In the elastic range

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{7}$$

which is constant. However, in the plastic range, the elements of the stress-strain matrix, D, are functions of the current stress components, as follows

$$D_{11} = 2G(1 - h_2 - 2h_1\,\sigma_x - h_3\,\sigma_x^2)$$

$$D_{22} = 2G(1 - h_2 - 2h_1\,\sigma_y - h_3\,\sigma_y^2)$$

$$D_{33} = 2G(1/2 - h_3\,\tau^2) \tag{8}$$

$$D_{12} = D_{21} = -2G(h_2 + h_1\{\sigma_x + \sigma_y\} + h_3\,\sigma_x\,\sigma_y)$$

$$D_{13} = D_{31} = -2G(h_1\,\tau + h_3\,\sigma_x\,\tau)$$

$$D_{23} = D_{32} = -2G(h_1\,\tau + h_3\,\sigma_y\,\tau)$$

where

$$h_1 = \frac{\left(\dfrac{3K}{G} \quad \dfrac{J_1}{6J_2^{1/2}}\right)}{J_2^{1/2}\left(1 + 9\alpha^2\dfrac{K}{G}\right)}$$

$$h_2 = \frac{\left(\alpha - \dfrac{J_1}{6J_2^{1/2}}\right)\left(\dfrac{3K}{G}\alpha - \dfrac{J_1}{6J_2^{1/2}}\right)}{\left(1 + 9\alpha^2\dfrac{K}{G}\right)} \; \frac{3K}{E} \; \frac{K}{J_2^{1/2}\left(1 + 9\alpha^2\dfrac{K}{G}\right)}$$

$$h_3 = \frac{1}{2J_2\left(1 + 9\alpha^2\dfrac{K}{G}\right)},$$

in which the explicit forms of the stress invariants are

$$J_1 = \sigma_x + \sigma_y + \sigma_z$$

$$J_2 = \frac{1}{6}(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + \tau^2.$$

The finite element method

Inasmuch as the derivation of element matrices involved in the finite element method are completely discussed elsewhere [3, 12 and 14], only the essential features and modifications will be mentioned herein.

The problem is set up for analysis by superposing a model consisting of triangular elements on an arbitrary portion of the extendend medium surrounding the cavity, as shown in Fig. 1 (for a circular cavity). Only one quadrant is considered here because of symmetry of geometry and loading (body forces are neglected). Within each element, the displacement field is assumed to be linear; hence the stress and strain fields in the element are constant. By interconnecting the triangles at the common nodes or vertices, continuity of displacements across common boundaries is automatically satisfied. It is then supposed that the model simulates the continuum sufficiently closely provided the triangular elements are made small enough.

The use of a non-uniform mesh of elements makes possible the distribution of smaller elements at locations where stress gradients are expected to be high. By using triangular elements, no difficulties arise from boundaries of irregular shape. It will usually be possible to generate all required matrices automatically from prescribed coordinates of boundary nodes and the desired distribution and number of elements, thereby avoiding much manual work [11].

The generation of the stiffness matrix of the system requires two transformation matrices for each element. If a generic element, m, has its vertices incident on nodes n_1, n_2, and n_3 with global coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3), respectively, the geometry of the element is specified by:

$$N_m = \frac{1}{x_{21}\,y_{31} - y_{31}\,y_{21}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{9}$$

where

$$x_{21} = x_2 - x_1, \quad y_{31} = y_3 - y_1, \quad \text{etc.}$$

The connection matrix of the element is described by a $(3 \times N)$ partitioned localization matrix, L_m, which contains a (2×2) identity matrix at positions $(1, n_1)$, $(2, n_2)$, and $(3, n_3)$, with zeroes elsewhere.

The stiffness matrix of the system is then given by

$$K = \sum_{m=1}^{M} A_m \left[L_m^t\, N_m^t\, D_m\, N_m\, L_m\right], \tag{10}$$

where A_m is the area of the element; M is the number of elements in the system.

The elements of D_m are given by Eq. 7 or 8 depending on whether the mth element is in the elastic or plastic state. Assuming that one or more of the elements are in the yielded state, it follows from Eq. 8 that the stiffness matrix of the system is a function of the stresses in the yielded elements. Therefore, the load must be applied in sufficiently small increments. The response of the model to each increment can then be determined by solving the equation of equilibrium of the stiffness method of analysis [4] which, for the νth stage of loading, can be written in the form

$$\Delta g^{(\nu)} = K^{(\nu-1)}\,\Delta_q^{(\nu)} \tag{11}$$

where $\Delta g^{(\nu)}$ and $\Delta q^{(\nu)}$ are, respectively, the load and displacement increment vectors, and $K^{(\nu-1)}$ is the stiffness matrix calculated from stresses at the $(\nu-1)$th increment.

The manner of support of the model fixes certain components of the displacement vector. For example, in Fig. 1, normal components of nodal displacements at the lines of symmetry are zero. Hence, for a prescribed load increment vector, the unknown components of the displacement increment vector can be solved from a subset of the system of equations in Eq. 11. In the present study, the system of equations was solved by the method of systematic over-relaxation [13]. Convergence to the solution is assured because the stiffness matrix corresponding to the subsystem of equations is positive definite [11].

The increments of strain in a generic element, m, can be calculated from the displacement increment vector by means of the transformation matrices of the element:

$$\Delta\varepsilon_m = N_m\,L_m\,\Delta q^{(\nu)} \tag{12}$$

To determine the new values of the stress components in the element, it is first assumed that the element deforms elastically. Therefore, the stresses are given by

$$\sigma_m^{(\nu)} = \sigma_m^{(\nu-1)} + D\,\Delta\varepsilon_m \tag{13}$$

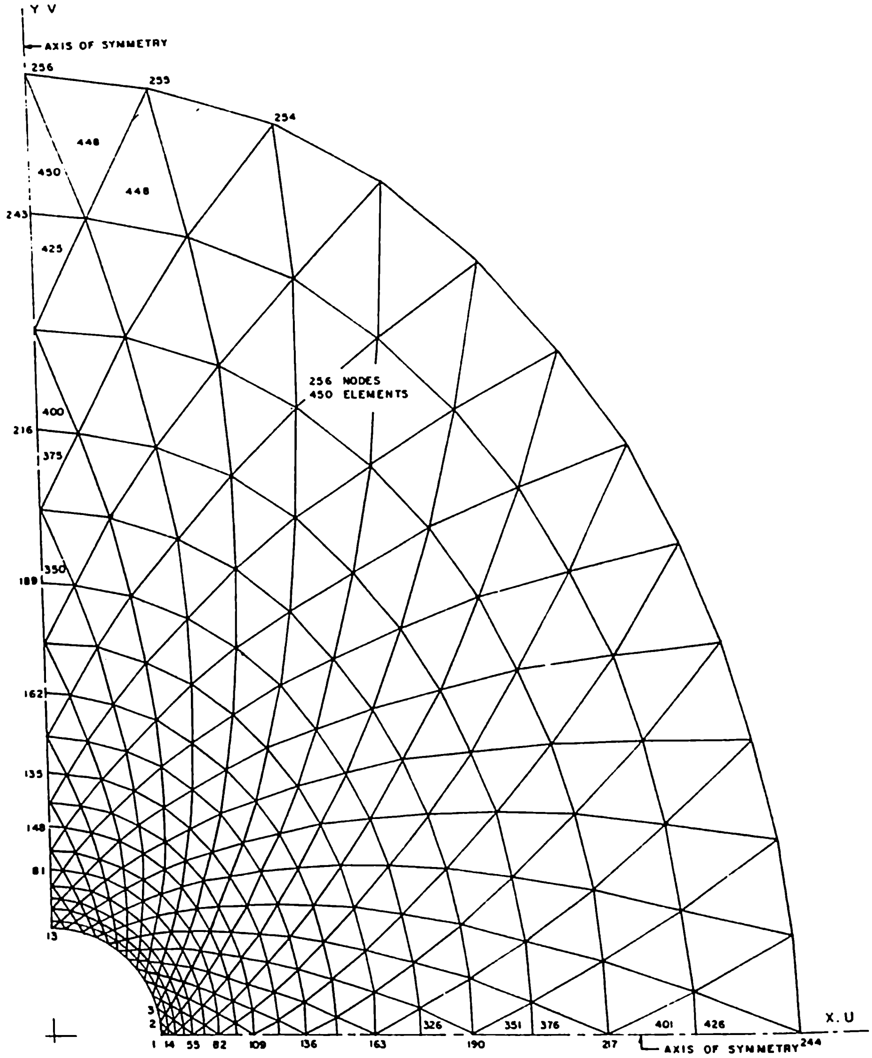


Fig. 1 — Finit element model for a circular cavity

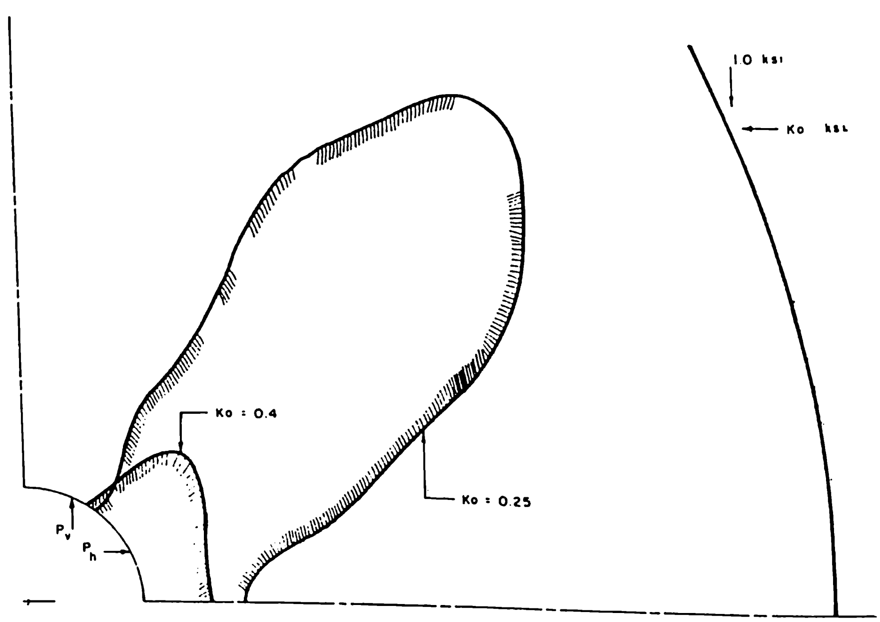


Fig. 2 — Plastic zones on complete unloading at the cavity wall

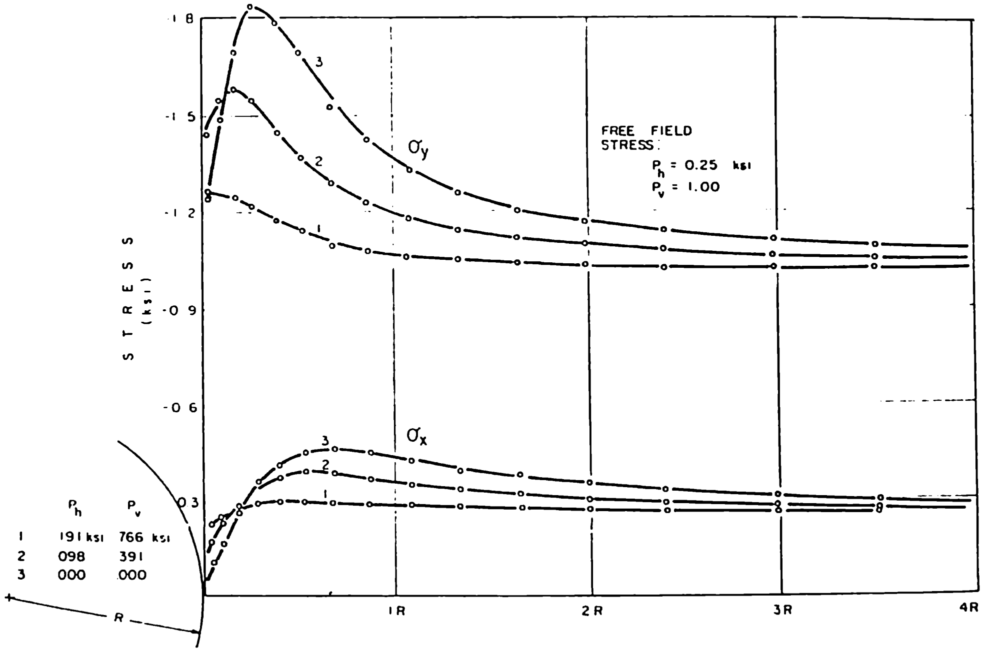


Fig. 3 — Stresses along horizontal section — $K_o = 0.25$

where D is the elastic stress-strain matrix (Eq. 7). The stresses are then substituted into the yield function, Eq. 1. If the criterion is satisfied, the stresses are recorded. If not, the stresses are recalculated, using Eq. 5, provided the given element was plastic during the previous load increment. In explicit form, Eq. 5 can be written as

$$
\begin{aligned}
\sigma_x{}^{(r)} &= \sigma_x{}^{(\nu-1)} + 2G\ \Delta\varepsilon_x - 2Ga - 2Gb\sigma_x{}^{(r-1)} \\
\sigma_y{}^{(r)} &= \sigma_y{}^{(\nu-1)} + 2G\ \Delta\varepsilon_y - 2Ga - 2Gb\sigma_y{}^{(r-1)} \\
\sigma_z{}^{(r)} &= \sigma_z{}^{(\nu-1)} - 2Ga - 2Gb\sigma_z{}^{(r-1)} \\
\tau^{(r)} &= \tau^{(\nu-1)} - G\gamma - 2Gb\tau^{(r-1)}
\end{aligned}
\qquad (14)
$$

and solved for stresses by successive substitutions [7]. The functions, a and b, are calculated from the stresses obtained at the $(r-1)$th cycle of substitutions. Usually three substitutions suffice.

If the element was previously elastic, the transition to the plastic state must be accounted for. Accordingly, a proportional part of the strain increment, $\Delta\varepsilon_m$ is used to calculate the elastic part of the stress increment such that the total stresses just satisfy the yield criterion. The remaining components of $\Delta\varepsilon_m$ are then used in Eq. 14 to determine the plastic part of the stress increments.

In general, the stresses calculated in the manner described will not satisfy equilibrium because of the linearization

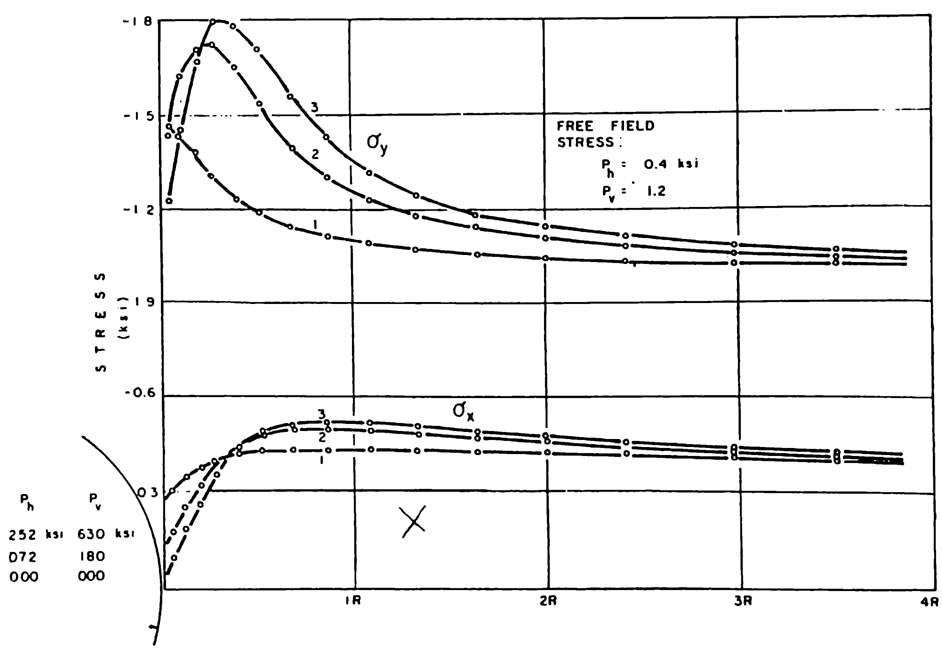


Fig. 4 — Stresses along horizontal section — $K_o = 0.4$

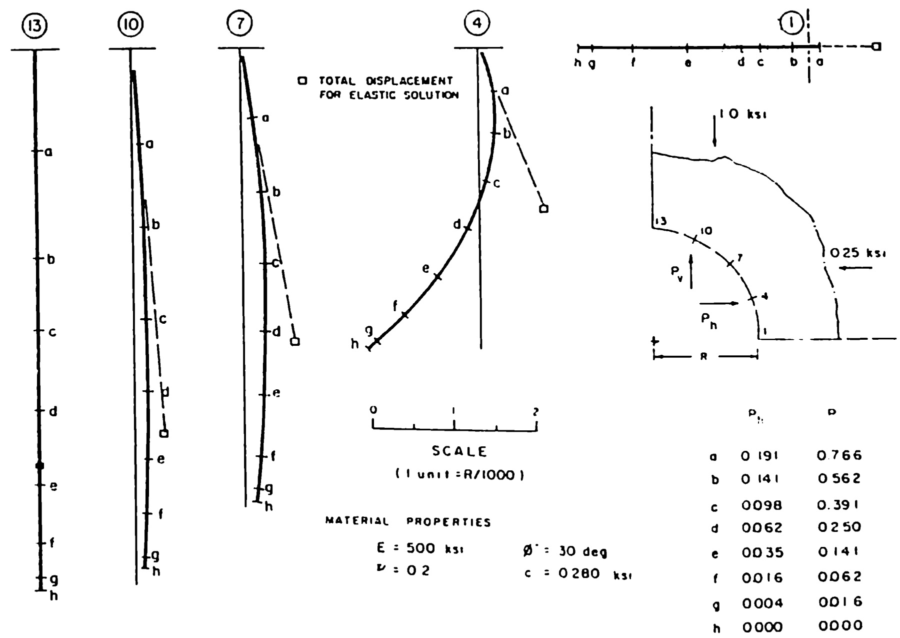


Fig. 5 — Displacement at typical points on face of cavity — $K_o = 0.25$

in Eq. 11. Therefore, an equilibrium check is made, as follows:

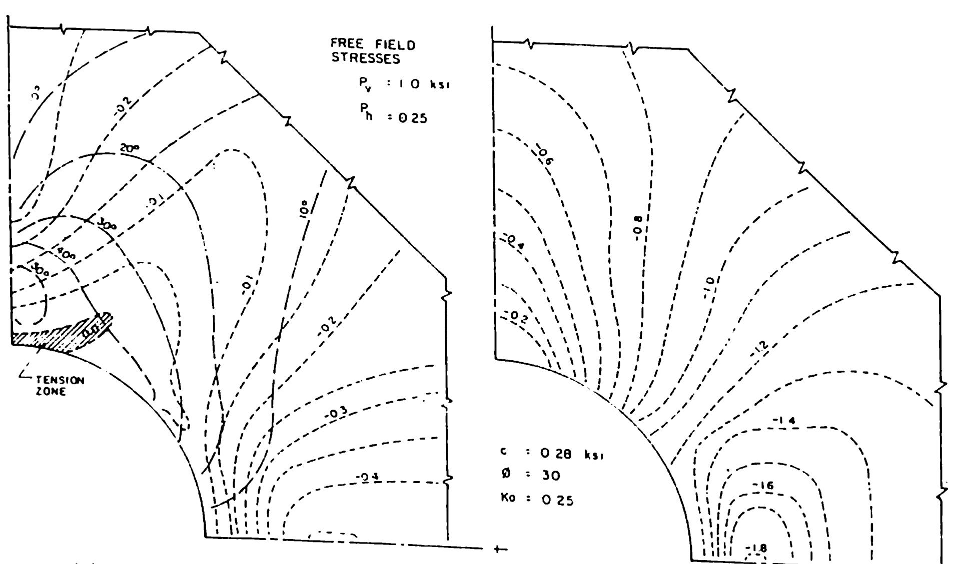
$$F = g^{(\nu)} - \sum_{m=1}^{M} A_m \left| L_m^t \, N_m^t \, \sigma_m^{(\nu)} \right| \tag{15}$$

Then, the unbalanced force vector, F, is used to compute for corrective nodal displacements, δq, using Eq. 11. The strain increment vector is then corrected by adding δq to it, and the process of obtaining the stresses followed by an equilibrium check is repeated. In the problems solved herein, three cycles of computations sufficed to make the unbalanced force components sufficiently small.

Illustrative problems

The problems solved in the present study were limited to circular openings for convenience, although the computer program has the capacity to treat openings of arbitrary shapes. Results of two problems, wherein the model shown in Fig. 1 was used, are briefly described. With a free field vertical stress of 1.0 ksi (kips per square inch), K_o values (ratio of horizontal to vertical free field stress) of 0.25 and 0.4 were used. Material properties were, $E = 500$ ksi, $\nu = 0.2$, $c = 0.280$ ksi, $\phi = 30$ degrees.

Computations were initialized by setting the stresses in the elements equal to the assumed free field values. Loads on the curved boundaries were then calculated to equilibrate these stresses. The model was «loaded» by stepwise reduction



(a) MAJOR PRINCIPAL STRESSES (ksi) AND DIRECTIONS (from horizontal)

(b) MINOR PRINCIPAL STRESSES

Fig. 6 — Principal stress contours — $K_o = 0.25$

of loads along the face of the cavity, in which horizontal and vertical loads were reduced proportionately.

Plastic zones on complete unloading are shown in Fig. 2, which illustrates very extensive yielding for K_o equal to 0.25. However, in both cases, the yield zone does not surround the cavity completely, on account of the low values of K_o used.

Normal and tangential stresses along the horizontal at various stages of loading are shown in Fig. 3 and 4. In both cases, the peak tangential stress occurs within the plastic zone. Fig. 5 shows the displacements of points along the face of the cavity for K_o equal to 0.25. Displacements for the elastic solution are also shown for comparison. As may be expected, yielding results in considerably greater displacements. Particularly notable are the large inward displacements of points opposite the yielded zone resulting from the expansion of the yielded material as predicted by the plastic potential theory for frictional materials [6].

Contours of principal stresses and principal stress directions in the vicinity of the cavity are shown in Fig. 6. Fig. 6(*a*) shows the extent of the tension zone at the crown of the cavity for $K_o = 0.25$. This zone can be verified to be smaller than that given by an elastic solution. In a solution using an angle of friction of 60 degrees (not presented here), no tension zone developed at the crown. These results suggest that the expansion of the plastic zone promotes a type of arching of the material above the crown of the cavity.

The foregoing results appear to be in general agreement with field observations and suggest that an elasto-plastic analysis can provide a possible description of inelasticity in rock.

Acknowledgements

This investigation is based on the first writer's doctoral dissertation, which was conducted at the University of Illinois under the supervision of Dr. D. U. Deere. All numerical work was done at the IBM 7094-1401 computer system of the university.

References

[1] Allen, D. N. de G. and R. Southwell — *Relaxation methods applied to engineering problems — XIV. Plastic straining in two dimensional stress systems*, Phil. Trans. Roy. Soc. A, vol. 242, pp. 379-414, 1950.

[2] Ang, A. H. S. and G. N. Harper — *Analysis of contained plastic flow in plane solids*, Proc. ASCE, vol. 84, No. EMS, pp. 397-418, 1964.

[3] Clough, R. W. — *The finite element method in plane stress analysis*, Proc. 2nd ASCE Conf. on Electronic Computation, pp. 345-378, 1960.

[4] De Veubeke, F. B. M. — *Displacement and equilibrium models in the finite element method*, in *Stress Analysis*, Zienciewicz, O. C. and G. S. Holister, Eds., Wiley, pp. 145-197, 1965.

[5] Drucker, D. C. — *Plasticity*, Proc. 1st Symp. on Naval Struc. Mech., Macmillan, pp. 407-455, 1958.

[6] Drucker, D. C., and W. Prager — *Soil mechanics and plastic analysis or limit design*, Quart. Appl. Math., vol. 10, No. 2, pp. 157-165, 1952.

[7] Hildebrand, F. B. — *Introduction to Numerical Analysis*, McGraw-Hill, 1956.

[8] Hill, R. — *Mathematical Theory of Plasticity*, Oxford U. P. 1950.

[9] Jacobs, J. A. — *Relaxation methods applied to problems of plastic flow*, Phil Mag. Ser. 7, vol. 41, pp. 349-361; 458-467, 1950.

[10] Pope, G. G. — *The application of the matrix displacement method in plane elastic-plastic problems*, Preprint, Proc. Conf. on Matrix Struct. Analysis, Dayton, Ohio, 1965.

[11] Reys, S. F. — *Elastic-Plastic analysis of underground openings by the finite element method*, Unpubl. Thesis, Univ. of Illinois, 1966.

[12] Sokolnikoff, I. S. — *Mathematical Theory of Elasticity*, McGraw-Hill, 1956.

[13] Young, D. W. — *The numerical solution of elliptic and parabolic equations*, in *Survey of Numerical Analysis*, J. Todd, Ed., McGraw-Hill, 1962.

[14] Wilson, E. L. — *Finite element analysis of two-dimensional structures*, Unpubl. thesis, Univ. of Cal., 1963.

"Elastic-Plastic Analysis of Underground Openings by the Finite Element Method"

S.F. Reyes and D.U. Deere

ERRATA

p. 478

Line 41:

$$\alpha = \frac{\tan \phi}{(9+12\tan^2\phi)^{1/2}} \qquad k = \frac{3c}{(9+12\tan^2\phi)^{1/2}} \tag{2}$$

Line 42:

In the elastic range ($f<k$ or $f=k$ and $\dot{f}<0$ in Eq. 1)

Line 50:

In the plastic range ($f=k$ and $\dot{f}=0$), perfect plasticity

Line 53:

$$\dot{\varepsilon}_{ij}^{(p)} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\left[\alpha\delta_{ij} + \frac{s_{ij}}{2J_2^{1/2}}\right]$$

Lines 64 to 66:

$$\dot{a} = \left(\dot{\varepsilon}_{\kappa\kappa} - \frac{3\alpha\dot{W}}{K}\right)\left[\left\{\frac{\frac{3K\alpha}{G} - \frac{J_1}{3J_2^{1/2}}}{1+9\alpha^2\frac{K}{G}}\right\}\left\{\alpha - \frac{J_1}{6J_2^{1/2}}\right\} - \frac{3\nu K}{E}\frac{k}{J_2^{1/2}(1+9\alpha^2\frac{K}{G})}\right] + \frac{\dot{W}}{k}\left[\alpha - \frac{J_1}{6J_2^{1/2}}\right]$$

$\dot{b} = \ldots$

Line 68:

$$\dot{W} = \sigma_{ij}\dot{\varepsilon}_{ij} \quad \text{and } K = \frac{E}{3(1-2\nu)}$$

Line 71: ____

$$\dot{\sigma} = D\dot{\varepsilon} \tag{6}$$

Line 72:

where $\dot{\sigma}^t = \left[\dot{\sigma}_x, \dot{\sigma}_y, \dot{\tau}\right]$ $\dot{\varepsilon}^t = \left[\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\gamma}\right]$

p. 479

Line 9:

$$h_2 = \frac{\left(\alpha - \dfrac{J_1}{6J_2^{1/2}}\right)\left(\dfrac{3K\alpha}{G} - \dfrac{J_1}{3J_2^{1/2}}\right)}{\left(1+9\alpha^2\frac{K}{G}\right)} - \frac{3\nu K}{E}\ \frac{k}{J_2^{1/2}\left(1+9\alpha^2\frac{K}{G}\right)}$$

Line 13:

$$J_2 = \frac{1}{6}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\right] + \tau^2$$

p. 481

Fig. 3:

P_h = -0.25 Ksi

P_v = -1.00 Ksi

Fig. 4:

P_h = -0.4 Ksi

P_v = -1.0 Ksi