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NONLINEAR EFFECTS OF GRAVITY ON EARTHQUAKE RESPONSE OF AN ELASTIC SHEAR-FLEXURAL BUILDING

Eric Augustus J. Tingatinga[1]* and Hideji Kawakami[2]

[1]*Institute of Civil Engineering*
University of the Philippines, Diliman, Quezon City
[2]*Geosphere Research Institute, Saitama University*
255 Shimo-Ohkubo, Sakura-ku, Saitama, 338-8570, Japan

ABSTRACT

An analytical building model that illustrates non-linear effects caused by gravity is introduced in this paper. Governing equations for the motion of one-story shear-flexural building subjected to earthquake-induced horizontal ground motion were developed taking into account large displacements. The response of typical structures subjected to harmonic ground excitation was expressed in exact and approximate formulations. Numerical examples show that large amplitude motion contains sub-harmonic components and increasing the amount of damping significantly decreases the higher mode contribution on the response. It was also shown that gravity generally decreases the natural frequency of elastic systems and that the apparent natural frequency further decreases with ground motion amplitude.

Keywords: shear-flexural building, gravity-effect, non-linearity

1. INTRODUCTION

The dynamics of structures is usually studied using shear building model that is based on the assumption that displacements are small and gravity effects are negligible. When structures are subjected to strong ground motions, however, large displacements are exhibited as a consequence of yielding, and gravity becomes the dominant force in causing the structure to collapse. In light of this fact, the authors propose a building model that can be used to study the nonlinear effects caused by gravity, and can describe more realistic response of structures undergoing large displacements during severe earthquake loading.

Several models, similar to that shown in Fig. 1, have been used to investigate the effect of gravity on the seismic response of flexural building models1-9. Jennings and Husid1 reported the increase in the natural period of elastic single-degree-of-freedom (SDOF) system compared to when gravity is ignored. Because of the assumption that yielding occurs when displacements or rotations are generally small, governing equations of motion are linearized and takes the form of Duffing equation. Duffing11 reported the jump phenomenon on amplitude response curves by studying a nonlinear differential equation that physically can be thought of as a forced vibration of a damped mechanical oscillator having a nonlinear spring. Although, the characteristics of motion of a flexural building model may be studied, according to the solution of Duffing equation, we note here that the solution is good only for relatively small values, say 15 degrees, of rotation angle. Hence, similar analytical procedure using perturbation methods will not be discussed in this paper.

*Correspondence to: Institute of Civil Engineering, University of the Philippines, Diliman, Quezon City, 1101, Philippines

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In pursuit of taking into account gravity force, the flexural building models mentioned above express the deformation of the column in terms of flexural (or rotational) deformation and assume that no shear deformation occurs. While it is difficult to distinguish between the shear and flexural rigidity of an actual building, a theoretical consideration is deemed important. This paper introduces a building model that can illustrate the coupled effect of shear and flexural rigidity on the response of a building undergoing large displacements. Seismic response analysis of typical buildings is presented in exact and approximate forms and solutions to governing equations are proposed. In order to investigate the effects of gravity on the dynamic properties of buildings, the shear-flexural building is compared with an equivalent flexure building and shear building models.

2. FORMULATION OF EQUATIONS

A simple building model that takes into account the effect of gravity is shown in Fig. 1. This model is subject to four assumptions: (1) the mass of the building is concentrated at the floor level; (2) the column of the building is inextensible axially; (3) the effects of axial force on the column's shear and flexural rigidity are negligible; and (4) shear deformations do not have influence on flexural rigidity, and vice versa. The path of the mass simplifies the complex horizontal and vertical motions of the deformed columns of a building. This model, herein named as a shear-flexural building, may exhibit significant vertical accelerations because of large displacements that are ignored in the shear building.

Considering the dynamic equilibrium of the mass and by using D'Alembert's principle we can derive the governing equations of this two-degree of freedom system, see Appendix. Expressed in terms of a generalized coordinate vector σ defined as

$$\sigma = \begin{Bmatrix} \delta \\ \theta \end{Bmatrix} \tag{1}$$

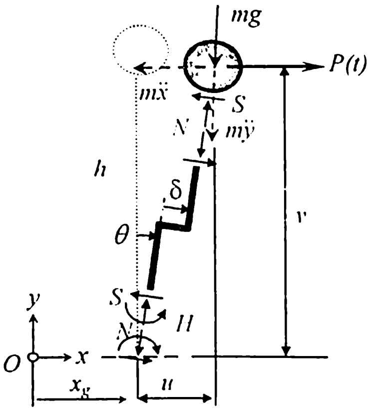

Fig. 1- One-story shear-flexural building subjected to lateral force $P(t)$ and earthquake-induced horizontal ground motion $x_g(t)$.

then we can write

$$M_\sigma \ddot{\sigma} + C\dot{\sigma} + F_\sigma = \left(-m\ddot{x}_g + P\right)\iota_\sigma \tag{2}$$

where

$$M_\sigma = \begin{bmatrix} m & mh \\ mh & m\left(h^2 + \delta^2\right) \end{bmatrix} \tag{3a}$$

$$C = \begin{bmatrix} c & 0 \\ 0 & \tilde{c} \end{bmatrix} \tag{3b}$$

$$F_\sigma = \begin{Bmatrix} k\delta - m\delta\dot{\theta}^2 - mg\sin\theta \\ \tilde{k}\theta + 2m\delta\dot{\delta}\dot{\theta} - mg(h\sin\theta + \delta\cos\theta) \end{Bmatrix} \tag{3c}$$

$$\iota_\sigma = \begin{Bmatrix} \cos\theta \\ h\cos\theta - \delta\sin\theta \end{Bmatrix} \tag{3d}$$

And when displacements are small, the equation reduces to

$$M\ddot{\sigma} + C\dot{\sigma} + K\sigma = \left(-m\ddot{x}_g + P\right)\iota \tag{4}$$

where

$$M = \begin{bmatrix} m & mh \\ mh & mh^2 \end{bmatrix} \tag{5a}$$

$$C = \begin{bmatrix} c & 0 \\ 0 & \tilde{c} \end{bmatrix} \tag{5b}$$

$$K = \begin{bmatrix} k & -mg \\ -mg & \tilde{k} - mgh \end{bmatrix} \tag{5c}$$

$$\iota = \begin{Bmatrix} 1 \\ h \end{Bmatrix} \tag{5d}$$

It should be noted that when the rotational stiffness $\tilde{k}$ is infinitely large, i.e. θ is very small, then Eq. (4) reduces to the governing equation of a shear building. And when k is large, the same equation reduces to the governing equation of the flexural building subject to small displacements,

$$mh^2\ddot{\theta} + \tilde{c}\dot{\theta} + \left(\tilde{k} - mgh\right)\theta = -mh\ddot{x}_g \tag{6}$$

The exact equation for the motion of a flexure building obtained as a special case of Eq. (16) is

$$mh^2\ddot{\theta} + \tilde{c}\dot{\theta} + \tilde{k}\theta - mgh\sin\theta = -mh\cos\theta\,\ddot{x}_g. \tag{7}$$

3. EARTHQUAKE RESPONSE

The governing equation for the motion of one-story shear-flexural building subjected to earthquake-induced horizontal ground motion is given by Eqs. (2) and (4) with P = 0. Response history analysis, however, cannot be easily performed since integration methods like Runge-Kutta-Gill12 cannot carried out directly because of the fact that the mass matrix $M\sigma$ involves linearly dependent rows. To be precise, row 2 is a multiple of row 1 in Eq. (3a) and in Eq. (5a) when δ equals zero, i.e., determinant is equal to zero and $M\sigma$ is singular. Here, we propose methods to obtain the response.

3.1 Exact solution

In order to examine the dynamic properties of the building undergoing large displacements, the motion associated with the solution of Eq. (2) must be studied. The second-order differential equations can be transformed into four first order differential equations by using a state vector

$$Y = \begin{Bmatrix} \sigma \\ \dot{\sigma} \end{Bmatrix} \tag{8}$$

whose time derivative can be written as

$$\dot{Y} = \begin{Bmatrix} \dot{\sigma} \\ M_\sigma^{-1} f_Y \end{Bmatrix}, \; f_Y = -m\ddot{x}_g(t)\iota_\sigma - C\dot{\sigma} - F_\sigma . \tag{9}$$

Since the linear system in terms of $\ddot{\sigma}$ can not be solved directly, i.e. M_σ^{-1} does not exist when $\delta = 0$ or δ is very small to cause significant loss in precision, we propose a method to obtain a smooth response. First, we use small Δt such that $\dot{Y}(t)$ can be approximated as

$$\dot{Y}(t) \cong \frac{Y(t+\Delta t) - Y(t)}{\Delta t} \tag{10}$$

And we can rewrite Eqs. (2) and (10) as

$$\begin{bmatrix} I & 0 \\ 0 & M_\sigma \end{bmatrix} \cdot Y(t+\Delta t) \approx \begin{Bmatrix} \sigma + \dot{\sigma}\Delta t \\ M_\sigma \dot{\sigma} + f_Y \Delta t \end{Bmatrix} \tag{11}$$

where I is a 2×2 identity matrix. Then, we introduce weight matrix W such that when δ becomes small, the following conditions are imposed

$$W\dot{\sigma}(t+\Delta t) \approx W\dot{\sigma}(t) \tag{12}$$

Here, W is a 2×2 diagonal matrix whose nonzero elements are the δ-dependent weights w_1 and w_2. Eqs. (11) and (12) form 6 linear equations in terms of 4 unknowns and can be written as

$$\begin{bmatrix} I & 0 \\ 0 & M_\sigma \\ 0 & W \end{bmatrix} \cdot Y(t+\Delta t) \cong \begin{Bmatrix} \sigma + \dot{\sigma}\Delta t \\ M_\sigma \dot{\sigma} + f_Y \Delta t \\ W\dot{\sigma} \end{Bmatrix} \tag{13}$$

Using the principle of least squares, we minimize the sum square of the errors

$$E = \left| f_Y - M_\sigma \ddot{\sigma}(t) \right|^2 + \left| W\ddot{\sigma}(t) \right|^2 \tag{14}$$

and the state vector $Y(t + \Delta t)$ can be solved giving us

$$Y(t+\Delta t) = \begin{Bmatrix} \sigma + \dot{\sigma}\Delta t \\ \dot{\sigma} + \left(M_\sigma^2 + W^2\right)^{-1} M_\sigma f_Y \Delta t \end{Bmatrix} \tag{15}$$

It is apparent from Eq. (15), that when W is zero, the solution obtained will be the same as the solution of Eq. (11). But when W is not zero, the difference will be minimized by selecting small values of weight functions w_1 and w_2 thereby enforcing Eq. (12). The weight function w_1 will be selected to be small and decreases exponentially with δ so as not to drastically modify Eq. (11). Here we propose the following functions:

$$w_1 = w_0 \exp\left(-\delta^2/\delta_c^2\right) \tag{16a}$$

$$w_2 = \rho \cdot w_1 h \tag{16b}$$

When using double precision, the smallest value of w_0 that can be used in floating point operations to obtain a nonzero determinant of $M_\sigma^2 + W^2$ when $\delta = 0$ is $(w_0)_{\min} = 10^{-8}mh$. The parameter δ_c controls the degree of flatness of w_1, so $(\delta_c)_{\min} = 10^{-4}h^{1.5}$ is the smallest value of δ_c that can be used to obtain a nonzero determinant when $w_1 \approx 0$. We suppose that small values of w_1 and w_2 will give smooth results and that when the amplitude of response is large, arbitrarily values of parameters w_0, δ_c and ρ may be used without causing significant effects on the steady-state response amplitude.

3.2 Small displacement approximations

Because of the aforementioned problem with the direct integration of Eq. (4), we will propose to obtain the steady-state response analytically. If, as an example, the structure is subjected to a harmonic ground motion

$$x_g(t) = x_{g0} \sin(\omega t) \tag{17}$$

then the particular solution of Eq. (4) can be assumed as

$$\sigma = A \sin(\omega t) + B \cos(\omega t) \tag{18}$$

where $A = \{A_1 \; A_2\}^T$ and $B = \{B_1 \; B_2\}^T$. Substituting Eq. (18) into Eq. (4), and equating coefficients of sine and cosine terms on both sides of the equation, we can solve the unknown constants A and B as

$$A = m\omega^2 x_{g0} \cdot \left[Y(\omega) + \omega^2 C Y(\omega)^{-1} C\right]^{-1} \iota \tag{19a}$$

$$B = -\omega \cdot Y(\omega)^{-1}(CA) \tag{19b}$$

Where $Y(\omega) = K - \omega^2 M$. Therefore, the response history of shear and flexural deformations can be expressed as

$$\delta(t) = \delta_0 \sin(\omega t - \phi_\delta) \tag{20}$$

$$\theta(t) = \theta_0 \sin(\omega t - \phi_\theta) \tag{21}$$

where

$$\delta_0 = \sqrt{A_1^2 + B_1^2}\,, \; \phi_\delta = \tan^{-1}(-B_1/A_1) \tag{22a}$$

$$\theta_0 = \sqrt{A_2^2 + B_2^2}\,, \; \phi_\theta = \tan^{-1}(-B_2/A_2), \tag{22b}$$

The relative and absolute displacement of the floor, and their corresponding phase, are therefore

$$u(t) = u_0 \sin(\omega t - \phi_r) \tag{23}$$

$$x(t) = x_0 \sin(\omega t - \phi_a) \tag{24}$$

Where

$$u_0 = \sqrt{(A_1 + hA_2)^2 + (B_1 + hB_2)^2} \tag{25a}$$

$$\phi_r = \tan^{-1}\left(-\frac{B_1 + hB_2}{A_1 + hA_2} \right) \tag{25b}$$

$$x_0 = \sqrt{(A_1 + hA_2 + x_{g0})^2 + (B_1 + hB_2)^2} \tag{25c}$$

$$\phi_a = \tan^{-1}\left(-\frac{B_1 + hB_2}{x_{g0} + A_1 + hA_2} \right) \tag{25d}$$

4. NUMERICAL ANALYSIS

To study the characteristics of response as well as to establish the reliability of the proposed numerical technique in the previous section, we investigate the response of an example of shear-flexural building whose mass m is 104 kg and height h is 5 m.

4.1 Effects of gravity on natural frequency

As with the previous flexural building models[1,2,4], we first take note of the effect of gravity on the natural frequency of the shear-flexural building and then compare it with the natural frequencies of the equivalent shear building and flexural building models. The natural frequency ω_{SFB} of shear-flexural building, obtained from the free-undamped vibration associated with Eq. (4), is

$$\omega_{SFB} = \sqrt{\frac{k(\tilde{k} - mgh) - (mg)^2}{m(\tilde{k} + mgh + h^2 k)}} \tag{26}$$

The subscript SFB is used to differentiate the natural frequency of the shear-flexural building with that of the shear building (SB) and flexural building (FB). The natural frequency of the shear-flexural building model is plotted, in Fig. 2, against flexural stiffness $\tilde{k}$ for different values of shear stiffness k and acceleration due to gravity g equal to 0 and 9.81 m/s^2. The natural frequency of the flexural building ω_{FB} with the same flexural stiffness, obtained from Eq. (6) and given here as

$$\omega_{FB} = \sqrt{\frac{\tilde{k} - mgh}{mh^2}} \tag{27}$$

can be interpreted as the intersection of the surface with g = 9.81 m/s^2 and the vertical plane parallel to ω_{SFB}-$\tilde{k}$ plane for a large value of k, e.g. plane defined by c-d-e. The corresponding plot for the natural frequency of the shear building, i.e., $\omega_{SB} = \sqrt{k/m}$, can be interpreted as the intersection of the surface with the vertical plane parallel to $\omega_{SFB} - k$ plane for a large value of $\tilde{k}$, e.g. surface defined by a-b. As shown in this plot, when $\tilde{k}$ is large the effect of gravity on natural frequency is negligible, hence the model is basically a shear building.

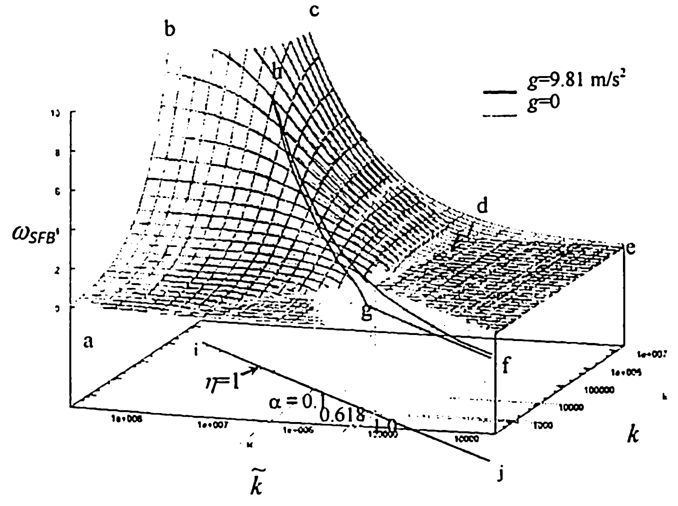

Fig. 2. Natural frequency of shear-flexural building model with $m = 10^4$ kg and $h = 5$ m.

We can observe from Fig. 2 that for any two plots with the same values $\tilde{k}$ and k, natural frequency is lesser when $g = 9.81$ m/s^2 than when $g = 0$. Hence, we can conclude that gravity generally decreases the natural frequency of the shear-flexural building. Furthermore, we can also observe that when g = 9.81 m/s^2, natural frequency is zero (or imaginary in a strict sense) for some values of $\tilde{k}$ and k as shown by surface a-g-d-e-f. But when $g = 0$, the natural frequency is always greater than zero which implies that the structure is always dynamically stable regardless of the value of $\tilde{k}$ and k. Hence, we can say that gravity may cause a shear-flexural building to become unstable depending on the values of $\tilde{k}$ and k. Eq. (26) suggests that the natural frequency is positive real number only for values of k and $\tilde{k}$ satisfying

$$\tilde{k} > mgh\,, \quad k > \frac{(mg)^2}{\tilde{k} - mgh} \tag{28}$$

The elastic stiffness will be approximated from a choice of gravity-effect parameter α given by

$$\alpha = \frac{mgh}{\tilde{k}} \tag{29}$$

This dimensionless parameter is similar to the parameter used by Sun, et.al.[2] to describe the increase in elastic natural period of flexure building due to gravity. Furthermore, if we assume that a building of given mass m and height h modeled as a shear building and a flexure building behave similarly when they undergo small displacements, then we can relate the shear and flexural stiffness, k and $\tilde{k}$, using a stiffness ratio η that we define here as

$$\eta = \frac{\tilde{k}}{h^2 k} \tag{30}$$

The natural frequency of the system, given by Eq. (26), can be expressed as

$$\omega_{SFB} = \sqrt{\frac{g}{h} \cdot \frac{1-\alpha-\alpha^2\eta}{\alpha(1+\eta+\alpha\eta)}} \tag{31}$$

When $\eta = 1$, the natural frequency is always positive for values of $\alpha < 0.618$. The natural frequency of the flexural building expressed in terms of α becomes

$$\omega_{FB} = \sqrt{\frac{\tilde{k}-mgh}{mh^2}} = \sqrt{\frac{g}{h} \cdot \frac{1-\alpha}{\alpha}} \tag{32}$$

And when $g = 0$ and $\eta = 1$, Eq. (31) reduces to the natural frequency of the shear building,

$$\omega_{FB} = \omega_{SB} = \sqrt{\frac{k}{m}} \tag{33}$$

Eqs. (31)-(33) correspond to surfaces h-g (for $\eta = 1$), c-e, and a-b in Fig. 2, respectively. As previously observed[1], the decrease in natural frequency when gravity is taken into account are shown by the plots corresponding to the flexural building and shear building (flexural building with $g = 0$ and $\eta = 1$). In other words, gravity causes the decrease in the value of natural frequency as shown by curves c-d-e and c-e. If the flexure building model exhibits shear deformations, the frequency further decreases with α as shown by corresponding plots for the shear-flexural building, e.g. curve g-h.

4.2 Characteristics of large-amplitude motion

The characteristics of response can be studied using the same shear-flexural building ($m = 10^4$ kg and $h = 5$ m) subjected to harmonic ground motion given in Eq. (17). We investigate the response as the building undergoes small and large displacements by using ground motion amplitudes x_{g0} equal to 0.2 m and 0.8 m, respectively. Here, we assume that the gravity-effect parameter defined by Eq. (29) is equal to $\alpha = 0.1$ and that the stiffness ratio η equal to 1. The damping ratio ξ defined as

$$\xi = \frac{c}{2m\omega_{SB}} = \frac{\tilde{c}}{2mh^2\omega_{FB}} \tag{34}$$

will be assumed equal to 2%, 5%, and 10%.

The steady-state response in terms of shear and flexural deformations, subject to small displacements assumptions will be obtained using Eqs. (20)-(22) and plotted with the exact solution. The exact solution using least squares will be computed using $\rho = 1$, $\delta_e = 0.1$ m, and $w_0 = 10^4, 10^2, 1$. The time histories of steady-state response will be shown in Fig. 3 for $t = 195$-200s.

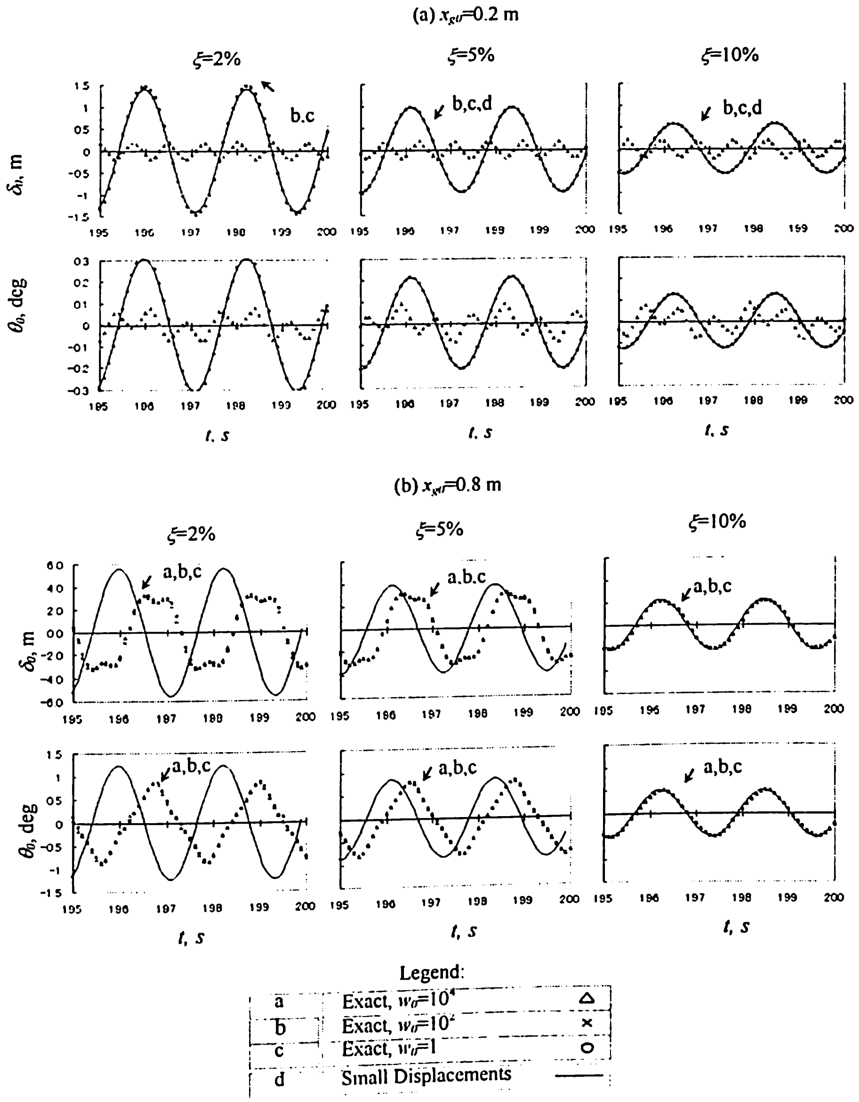

Fig. 3. Steady-state response history of shear-flexural building for (α = 0.1and η = 1) subjected to $x_g = x_{g0}$ sin(2.8t), where (a) x_{g0} = 0.2 m and (b) x_{g0} = 0.8 m. Exact solution computed using $w_0 = 10^4$, 10^2 and 1 (ρ = 1, δ_e = 0.1) and compared with the solution subject to small displacement assumptions.

We first establish the reliability of the numerical method proposed here by looking at the response shown in Fig. 3(a). This figure shows that when response is small, the value of parameter w_0 is important and must be selected to be relatively small. A large value will significantly affect the original equations, thereby resulting into a different motion which will be considered here as a numerical error, e.g. plots corresponding to $w_0 = 10^4$. The value of w_0 is insignificant when response amplitude increases as evidenced by the agreement of results for all values of w_0 used in Fig. 3(b). In the succeeding examples, a value of 1 will be used for w_0.

Now, with reference to Fig. 3, we take note of the characteristics of response of the building subjected to small and large amplitude ground motions. When displacements are small the response is harmonic with the same period as the ground motion, but when amplitude of response goes large the response contains sub-harmonic components. The amplitude, frequency and phase of the simple harmonics constituting the periodic large amplitude response can be obtained using Fourier series expansion. The Fourier amplitude plot for the response shown in Fig. 3(b) is shown in fig. 4. In this figure, the effect of damping on the contribution of higher modes on the response is apparent. Damping significantly decreases the amplitude of higher mode response, e.g. for shear deformation, changing the damping ratio from 2% to 5%, resulted to 25% decrease in amplitude of second mode compared to only 4% decrease in fundamental mode amplitude.

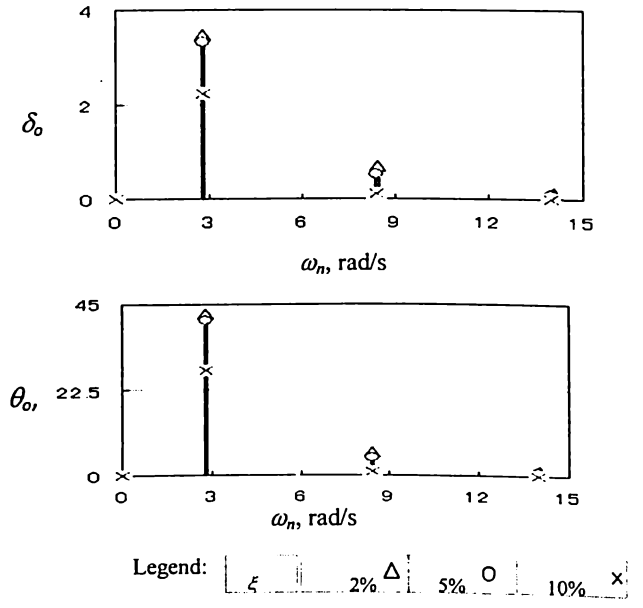

Fig. 4. Fourier amplitude plots corresponding to the responses to $x_g = 0.8\sin(2.8t)$ for 2%, 5%, and 10% damping ratios.

Finally, by considering only the amplitude of response corresponding to the fundamental mode, we can show in Fig. 5 the nonlinearity of response as amplitude of motion increases.

4.3 Effects of ground motion amplitude

It is a well-known fact that the actual natural frequency of buildings depends on the amplitude of ground motion. This amplitude-dependence cannot be observed in the shear building model or analytical models undergoing small displacements, but in nonlinear building models that exhibit large displacements, such as the flexure building whose equation is governed by Eq. (2), and of course, the shear-flexural building. We first note that for the flexural building, when Eq. (7) is transformed so that it includes the third-order term, solution to the corresponding Duffing equation suggests that the apparent natural frequency will increase with ground motion amplitude. This nonlinear stiffness is said to be of *hardening* type[11, 14].

To investigate how the natural frequency changes as ground motion amplitude increases, we compute the frequency response of the shear-flexural building subjected to harmonic excitation for increasing values of ground motion amplitude. Here we use, as an example, the same building model ($m = 10^4$ kg, $h = 5$ m, and $\alpha = 0.1$) subjected to ground motion amplitudes equal to 0.2 m, 1.0 m, and 2.0 m for damping ratios equal to 2%, 5%, and 10%.

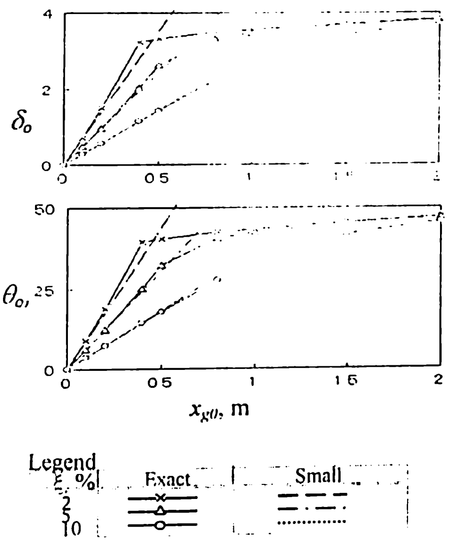

Fig. 5. Nonlinearity in steady-state response amplitude δ_0 and θ_0 using the fundamental mode.

The exact equation of motion of the system given in Eq. (2) will be solved numerically and the steady-state response will be estimated after several cycles. The response in terms of rotation angle (flexural deformation), shear deformation, relative and absolute displacements and their corresponding phase lags will be compared with the solution subject to small displacements assumption, i.e., Eqs. (23)-(25). The resulting frequency response curves corresponding to the fundamental mode are plotted in Fig. 6.

We observe from Fig. 6 that while the natural frequency of the shear-flexural building subject to small displacement assumptions does not change with ground motion amplitude, exact solutions show that the apparent natural frequency decreases with ground motion amplitude. Hence, in contrast to flexure building, the nonlinear effective stiffness is of *softening* type. We also conclude here that using practical values of forcing frequency ω (not completely shown in Fig. 6), the building exhibits no secondary resonance.

Finally, it is also interesting to note that when $x_{g0} = 2.0$ m and $\xi = 2\%$ and 5%, there is a sudden increase in response amplitude when forcing frequency is increased from 2.55 to 2.575 rad/s. This discontinuity is due to the superposition of ground displacement as evidenced by the phase reversal in absolute and relative displacement plots.

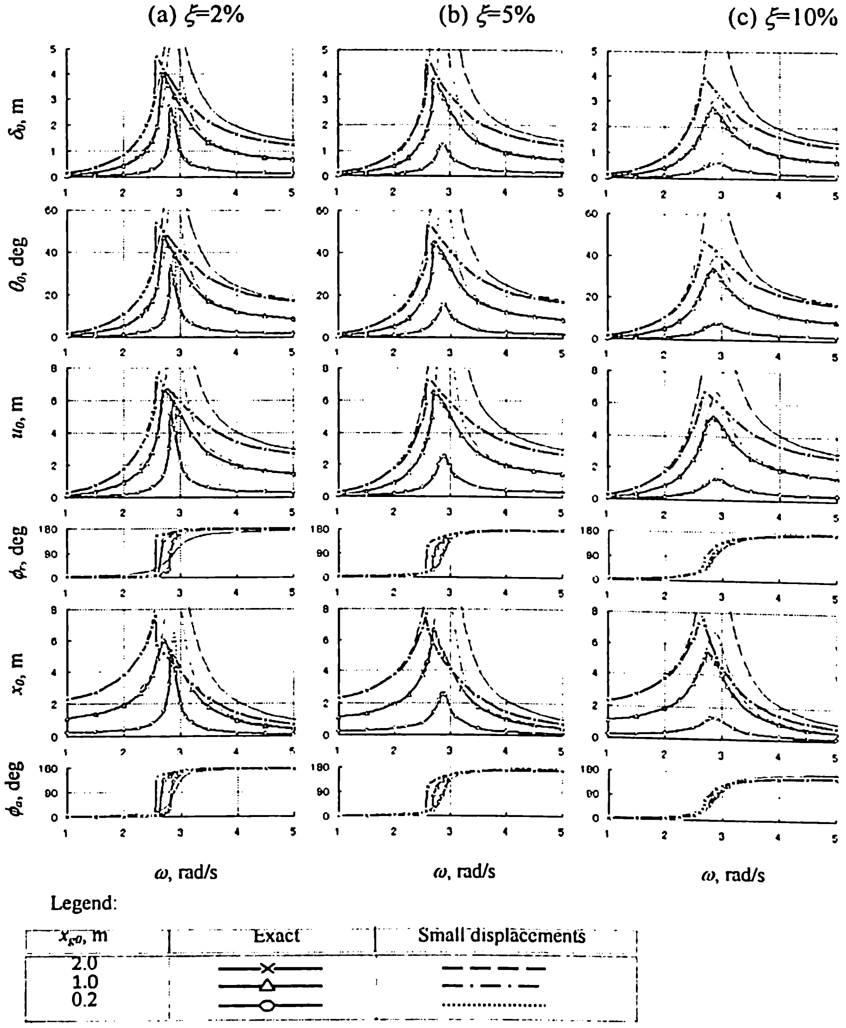

Fig. 6. Frequency response of SDOF shear-flexural building with $m = 10^4$ kg, $h = 5$ m, and gravity-effect parameter $\alpha = 0.1$ and $\eta = 1$. For all cases, the building is subjected to harmonic ground motion $x_g(t) = x_{g0}\sin(\omega t)$ at different values of ground motion amplitude x_{g0} equal to 0.2 m, 1.0 m, and 2.0 m. Damping ratio ξ is assumed to be 2%, 5% and 10%.

5. CONCLUSION

The fundamental equations governing the motion of an elastic shear-flexural building subjected to earthquake-induced ground motion and lateral loads are presented in this paper. Numerical examples of a one-story shear-flexural building subjected to large amplitude harmonic ground motion, show that the resulting motion contains sub-harmonic components and increasing the amount of damping significantly decreases the higher mode contribution on the response. It was also shown that gravity generally decreases the natural frequency, predicted by small displacement approximations, of elastic systems when compared with an equivalent shear building or flexure building. Exact solution shows that the apparent natural frequency further decreases with ground motion amplitude and that the nonlinear effective stiffness is of softening type.

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APPENDIX

Derivation of Motion Equations

Considering the dynamic equilibrium of the mass and by using D'Alembert's principle we can derive the governing equations of this two-degree of freedom system. With respect to the given *x-y* coordinate system shown in Fig. 1, the position of the mass can be expressed as

$$x = x_g + u\,,\ u = h\sin\theta + \delta\cos\theta \tag{A.1}$$

$$y = v = h\cos\theta - \delta\sin\theta \tag{A.2}$$

Differentiating these equations twice with respect to time we obtain the components of the acceleration as

$$\begin{aligned}\ddot{x} &= \frac{d}{dt}\left[\dot{x}_g + (h\cos\theta)\dot{\theta} - (\delta\sin\theta)\dot{\theta} + (\cos\theta)\dot{\delta}\right]\\ &= \ddot{x}_g + h\left(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta}\right) + \left(\cos\theta\,\ddot{\delta} - \sin\theta\,\dot{\delta}\dot{\theta}\right)\\ &\quad - \left[\delta\left(\cos\theta\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right) + (\sin\theta)\dot{\theta}\dot{\delta}\right]\\ &= \ddot{x}_g + v\ddot{\theta} + \cos\theta\,\ddot{\delta} - u\dot{\theta}^2 - 2\sin\theta\,\dot{\delta}\dot{\theta}\end{aligned} \tag{A.3}$$

$$\begin{aligned}\ddot{y} &= \frac{d}{dt}\left[-(h\sin\theta)\dot{\theta} - \left(\delta\cos\theta\dot{\theta} + \sin\theta\dot{\delta}\right)\right]\\ &= -h\left(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right) - \left(\cos\theta\,\dot{\theta}\,\dot{\delta} + \sin\theta\,\ddot{\delta}\right)\\ &\quad - \left[\delta\left(-\sin\theta\dot{\theta}^2 + \cos\theta\,\ddot{\theta}\right) + \cos\theta\,\dot{\theta}\dot{\delta}\right]\\ &= -u\ddot{\theta} - \sin\theta\,\ddot{\delta} - v\dot{\theta}^2 - 2\cos\theta\dot{\delta}\dot{\theta}\end{aligned} \tag{A.4}$$

Considering the dynamic equilibrium of the mass and by using D'Alembert's principle we can write,

$$N - mg\cos\theta - m\left(\ddot{x}\sin\theta + \ddot{y}\cos\theta\right) + P\sin\theta = 0 \tag{A.5}$$

$$S - mg\sin\theta - m\left(\ddot{y}\sin\theta - \ddot{x}\cos\theta\right) - P\cos\theta = 0 \tag{A.6}$$

$$H - Sh - N\delta = 0 \tag{A.7}$$

Subject to the conditions that the net restoring forces are

$$S = k\delta + c\dot{\delta} \tag{A.8}$$

$$H = \tilde{k}\theta + \tilde{c}\dot{\theta} \tag{A.9}$$

The governing equations of this two-degree of freedom system are obtained by solving S and H from Eqs. (A.6)-(A.7) then setting up conditions given in Eqs. (A.8) and (a.9). Substituting Eqs. (A.3) and (A.4) to Eqs. (A.5) and (A.6) and simplifying we have

$$\begin{aligned}N &= mg\cos\theta - P\sin\theta + m\left(\ddot{x}\sin\theta + \ddot{y}\cos\theta\right)\\ &= mg\cos\theta - P\sin\theta + m\left(v\sin\theta - u\cos\theta\right)\ddot{\theta}\\ &\quad - m\left(u\sin\theta + v\cos\theta\right)\dot{\theta}^2 - 2m\dot{\delta}\dot{\theta} + m\ddot{x}_g\sin\theta\\ &= mg\cos\theta - P\sin\theta - m\delta\ddot{\theta} - mh\dot{\theta}^2 - 2m\dot{\delta}\dot{\theta} + m\ddot{x}_g\sin\theta\end{aligned} \tag{A.10}$$

$$S = mg\sin\theta + P\cos\theta + m(\ddot{y}\sin\theta - \ddot{x}\cos\theta)$$
$$= mg\sin\theta + P\cos\theta + m(u\sin\theta + v\cos\theta)\ddot{\theta} - m\ddot{\delta}$$
$$- m(v\sin\theta - u\cos\theta)\dot{\theta}^2 - m\ddot{x}_g\cos\theta$$
$$= mg\sin\theta + P\cos\theta - mh\ddot{\theta} - m\ddot{\delta} + m\delta\dot{\theta}^2 - m\ddot{x}_g\cos\theta \quad (A.11)$$

Substituting Eqs. (A.10) and (A.11) into Eq. (A.8), we can solve for H as

$$H = \left(mg\sin\theta + P\cos\theta - mh\ddot{\theta} - m\ddot{\delta} + m\delta\dot{\theta}^2 - m\ddot{x}_g\cos\theta\right)h$$
$$+ \left(mg\cos\theta - P\sin\theta - m\delta\ddot{\theta} - mh\dot{\theta}^2 - 2m\dot{\delta}\dot{\theta} + m\ddot{x}_g\sin\theta\right)\delta$$
$$= mg(h\sin\theta + \delta\cos\theta) + P(h\cos\theta - \delta\sin\theta)$$
$$- m\left(h^2 + \delta^2\right)\ddot{\theta} - mh\ddot{\delta} - m(h\cos\theta - \delta\sin\theta)\ddot{x}_g - 2m\delta\dot{\delta}\dot{\theta} \quad (A.12)$$

Eqs. (A.11) and (A.12) can now be set-up using Eqs. (A.8) and (A.9) giving us,

$$mh\ddot{\theta} + m\ddot{\delta} - m\delta\dot{\theta}^2 + c\dot{\delta} + k\delta - mg\sin\theta = \left(-m\ddot{x}_g + P\right)\cos\theta\ , \quad (A.13)$$

$$m\left(h^2 + \delta^2\right)\ddot{\theta} + mh\ddot{\delta} + 2m\delta\dot{\delta}\dot{\theta} + \tilde{c}\dot{\theta} + \tilde{k}\theta - mg(h\sin\theta + \delta\cos\theta)$$
$$= (h\cos\theta - \delta\sin\theta)\left(-m\ddot{x}_g + P\right) \quad (A.14)$$

These equations govern the motion of a shear-flexural building and are written in compact form in Eq. (2).

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