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CONTINUOUS BEAMS ON ELASTIC SUPPORTS: AN ALTERNATIVE EQUATION

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ABSTRACT

The integrated equation of the elastic curve of an elastically supported continuous beam that is acted on by any number of concentrated forces, concentrated couples, and uniformly varying loads is obtained by using the Laplace transformation. The support may consist of a Winkler foundation, and/or simple supports and/or cantilever supports which may be linearly elastic or rigid.

INTRODUCTION

One of the earliest comprehensive treatment of the problem of beams on elastic foundation was made by Hetenyi [1] who obtained solutions to numerous problems involving infinite as well as finite length beams subjected to various systems of applied loads. Miranda and Nair [2] obtained the integrated equation of the elastic curve for arbitraiy loads when the beam is supported by a Winkler foundation only. Ting [3] obtained the equation for arbitrary loads when a single span beam is elastically retrained at the ends while resting on an elastic foundation.

In Ting's solution, expressions for the slope and deflection at the origin (left end of beam) are obtained by using two boundary conditions at the right end, namely,

$$-EIy''(L) = I_r y'(L)$$

and

$$-EIy'''(L) = K_r y(L)$$

where K_r and I_r are the deflection and rotation spring constants, respectively, for the right end support. The terms I_r and K_r, as well as I_1 and K_1 (the spring constants for the left support), are always present in Ting's solution regardless of whether the end restraints are elastic or rigid. These constants assume infinitive values when the end restraints are unyielding. This writer avoids the mathematical awkwardness of terms in the solution having infinite values for rigid supports by using a different approach in evaluating the deflection and the slope at the left end of the beam.

To make the solution free of any unevaluated integral, this paper considers only distributed loads that are uniformly varying since these are usually the loads of interest. In addition, to facilitate the writing of a computer program for determining deflections in single span as well as continuous beams, the presence of internal supports as well as end restraints is made explicit in the solution.

An application of the theory of beams on elastic foundation is the determination of the stresses in a thin-walled circular cylinder that is acted on by rotationally symmetric loads [1], [4]. The cylinder may be provided with circumferential stiffener rings. When the stiffener rings are equally spaced and the load is uniformly distributed along the cylinder, a longitudinal strip of the cylinder between two consecutive rings can be treated as a single span beam on elastic foundation that is cantilevered at both ends. [1]. If for some construction or other reasons the stiffener rings can not be spaced equally, a longitudinal strip of the cylinder behaves as a continuous beam on an elastic foundation. The integrated equation of the elastic curve obtained in this paper solves this and other problems. The solution obtained includes the results in [2] and [3] as special cases.

Another application of the solution obtained in this paper is the analysis of beams that are elastically supported by perpendicular cross beams.

Derivation of the Equation:

The differential equation of the elastic curve of a beam on an elastic foundation (Winkler foundation) is

$$EI\frac{d^4y}{dx^4} + ky = q(x) \tag{1}$$

where
- $y(x)$ = beam deflection
- EI = flexural rigidity of the beam
- k = modulus of the foundation
- $q(x)$ = load function for the beam, i.e., the intensity of the distributed load at a point x in the beam.

When the loads on the beam consist of concentrated forces, concentrated couples, and uniformly varying distributed loads, and if the beam is additionally supported by cantilevers as well as simple supports, the free body diagram of the beam will be as shown in Figure 1. For convenience, the origin of coordinates is located slightly to the left of the leftmost load or reaction. Thus, the leftmost loads or reaction will be at $x = O+$.

L is the length of the beam, c_i, a_i-b_i, d_i, g_i, and f_i are the x coordinates respectively of n concentrated forces P_i, m distributed loads A_i, B_i, j applied concentrated couples Z_i, the location

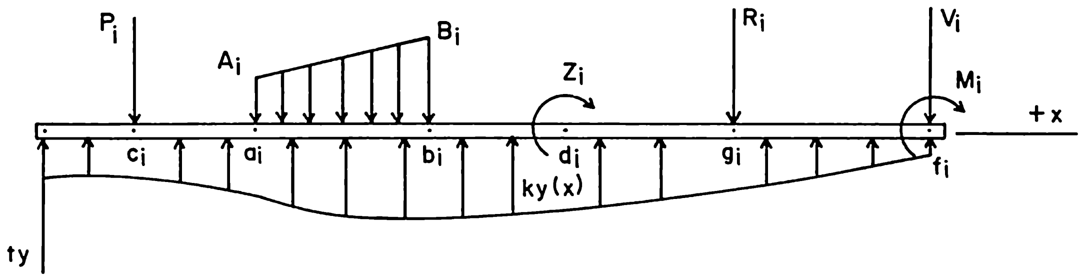


Fig. 1

of h simple supports, and the location of p (p=O, 1 or 2) cantilevers. R_i is the reaction at a simple support. M_i and V_i are the reactions at a cantilever support which may only be located at the left and/or the right end of the beam. The upward distributed force ky(x) represents the reaction of the foundation.
Let $k = 4\beta^4 EI$. Then,

$$\frac{d^4 y}{dx^4} + 4\beta^4 y = \frac{q(x)}{EI} \tag{2}$$

The load functions for the applied forces and the support reactions are (see Pacheco [5])

for P_i : $\sum_{i=1}^{n} P_i \delta(x - c_i)$

for Z_i : $-\sum_{i=1}^{j} Z_i \delta'(x - d_i)$

for A_i-B_i : $\sum_{i=1}^{m} \left[A_i + \frac{B_i - A_i}{b_i - a_i}(x - a_i) \right] \left[u(x - a_i) - u(x - b_i) \right]$

for R_i : $\sum_{i=1}^{h} R_1 \delta(x - g_i)$

for V_i : $\sum_{i=1}^{p} V_i \delta(x - f_i)$

for M_i : $-\sum_{i=1}^{p} M_i \delta'(x - f_i)$

where u is the unit step function, δ is the Dirac delta function, and δ' is the unit doublet.

If a simple support is linearly elastic, R_i is to be replaced by $-k_i y(g_i)$ where k_i is the spring constant of the simple support and $y(g_i)$ is the unknown deflection at that support. Similarly, at a linearly yielding cantilever, V_i is to be replaced by $-K_i y(f_i)$ and M_i is to be replaced by $-H_i y'(f_i)$, where K_i and H_i are the deflection and rotation spring constants, respectively.

Substitute the load function into (2) to get

$$\mathrm{EI}\left[\frac{d^4 y}{dx^4} + 4\beta^4 y\right] = \sum_{i=1}^{n} P_i \delta\left(x - c_i - \sum_{i=1}^{l} Z_i \delta'(x - d_i)\right)$$

$$+\sum_{i=1}^{m}\left[A_i + \frac{B_i - A_i}{b_i - a_i}(x - a_i)\right]\left[u(x - a_i) - u(x - b_i)\right]$$

$$+\sum_{i=1}^{1} R_i \delta(x - g_i) + \sum_{i=1}^{p} V_i \delta(x -- f_i) - \sum_{i=1}^{p} M_i \delta'(x - f_i) \qquad (3)$$

Take the Laplace transform of both sides of (3) to obtain

$$L\{y\} = \frac{s^3 y(0)}{s^4 + 4\beta^4} + \frac{s^2 y'(0)}{s^4 + 4\beta^4} + \frac{s y''(0)}{s^4 + 4\beta^4} + \frac{y'''(0)}{s^4 + 4\beta^4}$$

$$+\sum_{i=1}^{n} \frac{P_i}{EI} \frac{\exp(-c_i s)}{s^4 + 4\beta^4} - \sum_{i=1}^{l} \frac{Z_i}{EI} \frac{s\exp(-d_i s)}{s^4 + 4\beta^4} + \sum_{i=1}^{m} \frac{1}{EI(s^4 + 4\beta^4)}$$

$$\left.\frac{A_i}{s} + \frac{B_i - A_i}{b_i - a_i}\frac{1}{s^2}\right]\left[\exp(-a_i s) - \exp(-b_i s)\right]$$

$$+\sum_{i=1}^{1} \frac{R_i}{EI} \frac{\exp(-g_i s)}{s^4 + 4\beta^4} + \sum_{i=1}^{p} \frac{V_i}{EI} \frac{\exp(-f_i s)}{s^4 + 4\beta^4} - \sum_{i=1}^{p} \frac{M_i}{EI} \frac{s\exp(-f_i s)}{s^4 + 4\beta^4} \qquad (4)$$

Since the origin was placed slightly to the left of the leftmost load and reaction, then

$$y''(0) = \frac{M(0)}{EI} = 0 \quad \text{and} \quad y'''(0) = \frac{V(0)}{EI} = 0$$

where M(0) and V(0) are the shea.· force and bending moment, respectively, at x = 0.

From Erdelyi [6],

$$L^{-1}\left\{4\beta^{3}\frac{as^{3}+bs^{2}+cs+d}{s^{4}+4\beta^{4}}\right\}=4\beta^{4}a\cos(\beta x)\cosh(\beta x)+$$

$$(2\beta^{2}b-d)\cos(\beta x)\sinh(\beta x)+(2\beta^{2}b+d)\sin(\beta x)\cosh(\beta x)+$$

$$2\beta c\sin(\beta x)\sinh(\beta x)$$

Using the convolution integral in getting the inverse transform of the term involving the distributed loads, the inverse transform of (4) is found to be

$$y(x)=y(0)\Phi_0(\beta x)+\frac{1}{\beta}y'(0)\Phi_1(\beta x)+\sum_{i=1}^{n}\frac{P_i}{\beta^{3}EI}\Phi_3[\beta(x-c_1)]u(x-c_i)$$

$$+\frac{1}{4\beta^{3}EI}\sum_{i=1}^{m}\left[\left[\left\{\frac{A_i}{\beta}\left(1-\Phi_0[\beta(x-a_i)]\right)+\frac{B_i-A_i}{b_i-a_1}\left[-\frac{\Phi_1[\beta(x-a_i)]}{\beta^{2}}+\frac{x-a_i}{\beta}\right]\right\}\right]\right]$$

$$u(x-a_i)-\left\{\frac{A_i}{\beta}\left(1-\Phi_0[\beta(x-b_i)]+\frac{B_i-A_i}{b_i-a_i}\left[-\frac{\Phi_1[\beta(x-b_i)]}{\beta^{2}}\right]\right.\right.$$

$$\left.\left.+\frac{a_i-b_i}{\beta}\Phi_0[\beta(x-b_i)]+\frac{x-a_1}{\beta}\right)\right\}$$

$$u(x-b_i)-\sum_{i=1}^{j}\frac{Z_i}{\beta^{2}EI}\Phi_2[\beta(x-d_i)]u(x-d_i)+\sum_{i=1}^{l}\frac{R_i}{\beta^{3}EI}$$

$$\Phi_3[\beta(x-g_i)]u(x-g_i)+\sum_{i=1}^{p}\frac{V_i}{\beta^{3}EI}\Phi_3[\beta(x-g_i)]u(x-g_i)$$

$$-\sum_{i=1}^{p}\frac{M_i}{\beta^{2}EI}\Phi_2[\beta(x-f_i)]u(x-f_i), \qquad (5)$$

where, following the notation of Miranda and Nair [2],

$$\Phi_0(\beta x) = \cos(\beta x)\cosh(\beta x)$$

$$\Phi_1(\beta x) = \frac{1}{2}\left[\sin(\beta x)\cosh(\beta x) + \cos(\beta x)\sinh(\beta x)\right]$$

$$\Phi_2(\beta x) = \frac{1}{2}\sin(\beta x)\sinh(\beta x)$$

$$\Phi_3(\beta x) = \frac{1}{4}\left[\sin(\beta x)\sinh(\beta x) - \cos(\beta x)\sinh(\beta x)\right]$$

Two equations of equilibrium yield the following:

$$\sum F = 0 = \sum_{i=1}^{n} P_i + \sum_{i=1}^{m} \frac{A_i + B_i}{2}(b_i - a_i) + \sum_{i=1}^{1} R_i + \sum_{i=1}^{p} V_i - k\int_o^L y(x)dx \tag{6}$$

$$\sum M_0 = 0 = \sum_{i=1}^{n} P_i c_i + \sum_{i=1}^{m}\left[\frac{A_i}{2}(b_i^2 - a_i^2) + \frac{(B_i - A_i)(b_i - a_i)(a_i + 2b_i)}{6}\right]$$

$$+\sum_{i=1}^{j} Z_i + \sum_{i=1}^{1} R_i g_i + \sum_{i=1}^{p} V_i f_i + \sum_{i=1}^{p} M_i - k\int_o^L xy(x)dx \tag{7}$$

where y(x) is as given by (5).

Through routine integration, it can be shown that

$$\int \Phi_0(\beta x)dx = \Phi_1(\beta x) / \beta + c$$

$$\int \Phi_1(\beta x)dx = \Phi_2(\beta x) / \beta + c$$

$$\int \Phi_2(\beta x)dx = \Phi_3(\beta x) / \beta + c$$

$$\int \Phi_3(\beta x)dx = -\Phi_0(\beta x / 4\beta) + c$$

$$\int x\Phi_0(\beta x)dx = x\Phi_1(\beta x)/\beta - \Phi_2(\beta x)/\beta^2 + c$$

$$\int x\Phi_1(\beta x)dx = -\Phi_3(\beta x)/\beta^2 + x\Phi_2(\beta x)/\beta + c$$

$$\int x\Phi_2(\beta x)dx = x\Phi_3(\beta x)/\beta + \Phi_0(\beta x)4\beta^2 + c$$

$$\int x\Phi_3(\beta x)dx = \Phi_1(\beta x)/4\beta^2 - x\Phi_0(\beta x)/4\beta + c$$

Use of the above integration formulas in evaluating $\int y(x)sx$ and $\int xy(x)dx$ results in

$$A + \sum_{i=1}^{l} R_i C_i + \sum_{i=1}^{p} V_i D_i + \sum_{i=1}^{p} M_i F_i - Gy(0) - Hy'(0) = 0 \tag{8}$$

and

$$J + \sum_{i=1}^{l} R_i N_i + \sum_{i=1}^{p} V_i Q_i + \sum_{i=1}^{p} M_i S_i - Ty(0) - Uy'(0) = 0 \tag{9}$$

where

$$A = \sum_{i=1}^{n} P_i\Phi_0[\beta(L-c_i)] + \sum_{i=1}^{m}\left[\frac{A_i + B_i}{2}(b_i - a_i) - A_i\left(L - a_i - \Phi_1[\beta(L-a_i)]/\beta\right)\right.$$

$$+\frac{B_i - A_i}{b_i - a_i}\left(\Phi_2[\beta(L-a_i)]/\beta^2\right) + A_i\left(L - b_i - \Phi_1[\beta(L-b_i)]/\beta\right)$$

$$+\frac{B_i - A_i}{b_i - a_i}\left\{-\Phi_2[\beta(L-b_i)]/\beta^2 + (a_i - b_i)\Phi_1[\beta(L-b_i)]/\beta - \frac{(b_i - a_i)^2}{2}\right\}$$

$$+\sum_{i=1}^{l}\frac{kZ_i}{\beta^3 EI}\Phi_3[\beta(L-d_i)]$$

$$C_i = \Phi_0[\beta(L-g_i)], \quad D_i = \Phi_0[\beta(L-f_i)], \quad F_i = k\Phi_3[\beta(L-f_i)]/\beta^3 EI$$

$$G = k\Phi_1(\beta L)/\beta, \quad H = k\Phi_2(\beta L)/\beta^2$$

$$J = \sum_{i=1}^{n} P_i\left[-\Phi_1\left[\beta\left(L-c_i\right)\right]/\beta + L\Phi_0\left[\beta\left(L-c_i\right)\right]\right] + \sum_{i=1}^{m}\left[\frac{A_i}{2}\left(b_i^{\,2}-a_i^{\,2}\right)+\right.$$

$$\frac{\left(B_i - A_i\right)\left(b_i - a_i\right)\left(a_i + 2b_i\right)}{6} - A_i\left\{\left(L^2 - a_i^{\,2}\right)/2 - L\Phi_1\left[\beta\left(L-a_i\right)\right]/\beta\right.$$

$$+\Phi_2\left[\beta\left(L-a_i\right)\right]/\beta^2\right\} + \frac{B_i - A_i}{b_i - a_i}\left[L\Phi_2\left[\beta\left(L-a_i\right)\right]/\beta^2 - \Phi_3\left[\beta\left(L-a_i\right)\right]/\beta^3\right.$$

$$-\left(2L^3 - 3a_iL^2 + a_i^{\,3}\right)/6] + A_i\left\{\left(L^2 - b_i^{\,2}\right)/2 - L\Phi_1\left[\beta\left(L-b_1\right)\right]/\beta\right.$$

$$-\left(2L^3 - 3a_iL^2 + a_i^{\,3}\right)/6] + A_i\left\{\left(L^2 - b_1^{\,2}\right)/2 - L\Phi_1\left[\beta\left(L-b_1\right)\right]/\beta\right.$$

$$+\Phi_2\left[\beta\left(L-b_i\right)\right]/\beta^2\right\} + \frac{B_i - A_i}{b_i - a_i}\left[-L\Phi_2\left[\beta\left(L-b_i\right)\right]/\beta^2 + \Phi_3\left[\beta\left(L-b_i\right)\right]/\beta\right.$$

$$+\left(a_i - b_i\right)\left\{L\Phi_1\left[\beta\left(L-b_i\right)\right]/\beta\right\} - \Phi_2\left[\beta\left(L-b_i\right)\right]/\beta^2\}$$

$$-\left(2L^3 - 3a_iL^2 - 2b_i^{\,3} + 3a_ib_i^{\,2}\right)/6]$$

$$+\sum_{i=1}^{j}\frac{kZ_i}{\beta^3EI}\left\{L\Phi_3\left[\beta\left(L-d_i\right)\right] + \Phi_0\left[\beta\left(L-d_i\right)\right]/4\beta\right\}$$

$$N_i = L\Phi_0\left[\beta\left(L-g_i\right)\right] - \Phi_1\left[\beta\left(L-g_i\right)\right]/\beta,\quad Q_i = L\Phi_0\left[\beta\left(L-f_i\right)\right] - \Phi_1\left[\beta\left(L-f_i\right)\right]/\beta$$

$$S_i = \frac{k}{\beta^3EI}\left\{L\Phi_3\left[\beta\left(L-f_i\right)\right] + \Phi_0\left[\beta\left(L-f_i\right)\right]/4\beta\right\}$$

$$T = k\left[L\Phi_i(\beta L)/\beta - \Phi_2(\beta L)/\beta^2\right],\qquad \mathrm{U} = k\left[\Phi_2(\beta L)/\beta^2 - \Phi_2(\beta L)/\beta^3\right]$$

Solve (8) and (9) simultaneously for y(0) and y'(0) to obtain

$$y(0) = \frac{AU - HJ + \sum_{i=1}^{1} R_i\left(UC_i - HN_i\right) + \sum_{i=1}^{p} V_i\left(UD_i - HQ_i\right) + \sum_{i=1}^{p} M_i\left(UF_i - HS_i\right)}{GU - HT}$$

$$y'(0)=\frac{GJ-AT+\sum_{i=1}^{1}R_i(GN_i-TC_i)+\sum_{i=1}^{p}V_i(GQ_i-TD_i)+\sum_{i=1}^{p}M_i(GS_i-TF_i)}{GU-HT}$$

Substitute the expressions for y(0) and y'(0) into (5) to get the integrated equation of the elastic curve:

$$y(x)=\frac{AU-HJ}{GU-HT}\Phi_0(\beta x)+\frac{(GJ-AT}{GU-HT}\frac{\Phi_1(\beta x)}{\beta}$$

$$+\sum_{i=1}^{n}\frac{P_i}{\beta^3 EI}\Phi_3[\beta(x-c_i)]u(x-c_i)-\sum_{i=1}^{j}\frac{Z_i}{\beta^2 EI}\Phi_2[\beta(x-d_i)]u(x-d_i)$$

$$+\frac{1}{4\beta^3 EI}\sum_{i=1}^{m}\left[\frac{A_i}{\beta}\left(1-\Phi_0[\beta(x-a_i)]+\frac{B_i-A_i}{b_i-a_i}\right)\left[-\frac{\Phi_1[\beta(x-a_i)]}{\beta^2}\right]\right.$$

$$+\frac{x-a_i}{\beta}\ ]\ \}u(x-a_i)-\frac{A_i}{\beta}(1-\Phi_0[\beta(x-b_i)]$$

$$+\frac{B_i-A_i}{b_i-a_i}\left[\frac{\Phi_1[\beta(x-b_i)]}{\beta^2}-\frac{a_i b_i}{\beta}\Phi_0[\beta(x-b_i)]\right]$$

$$\left.-\frac{x-a_i}{\beta}\ ]u(x-b_i)\right]+\sum_{i=1}^{1}R_i[\Phi_3[\beta(x-g_i)]u(x-g_i)/\beta^3 EI$$

$$\left.+\frac{UC_i-HN_i}{GU-HT}\Phi_0(\beta x)+\frac{GN_i-TC_i}{GU-HT}\frac{\Phi_1(\beta x)}{\beta}\right]$$

$$+\sum_{i=1}^{p}V_i[\Phi_3[\beta(x-g_i)]u(x-g_i)/\beta^3 EI$$

$$\left.+\frac{UD_i-HQ_i}{GU-HT}\Phi_0(\beta x)+\frac{GQ_i-TD_i}{GU-HT}\frac{\Phi_1(\beta x)}{\beta}\right]$$

$$-\sum_{i=1}^{p}M_i[\Phi_2[\beta(x-f_i)]u(x-f_i)/\beta^2 EI$$

$$+\frac{UF_i - HS_i}{GU - HT}\Phi_0(\beta x) + \frac{GS_i - TF_i}{GU - HT}\frac{\Phi_1(\beta x)}{\beta} \tag{10}$$

The expressions for the slope, shear, and bending moment can be found by differentiating (10). The following differentiation formulas from Miranda and Nair [2] will be helpful:

Table 1
Differentiation Formulas for Φ_n

Φ_n	Φ_n'	Φ_n''	Φ_n'''
Φ_0	$-4\beta\Phi_3$	$-4\beta^2\Phi_2$	$-4\beta^3\Phi_1$
Φ_1	$\beta\Phi_0$	$-4\beta^2\Phi_3$	$-4\beta^3\Phi_2$
Φ_2	$\beta\Phi_1$	$\beta^2\Phi_0$	$-4\beta^3\Phi_3$
Φ_3	$\beta\Phi_2$	$\beta^2\Phi_1$	$\beta3\Phi_0$

DISCUSSION OF RESULTS

Equation (10) involved 1 unknown reactions R_i and, if the beam is also cantilevered, 2p unknown reactions at the cantilever, where p is the number of cantilever supports (p = 0, 1 or 2). The values of these reactions can be found by using the fact that the deflection at a simple unyielding support is zero and that $y'(f_i) = 0$, where f_i is the location of the cantilever.

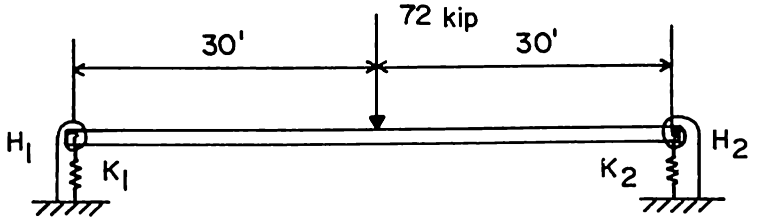
If the simple support at g_i is an elastic spring with spring constant k_i, the reaction R_i in (10) will be replaced by $-k_i y(g_i)$. Hence, there is available from (10) one equation for each elastic support, thus enabling the determination of the deflection $y(g_i)$.

If the cantilevers are elastic, then the reaction Vi will be replaced by $-K_i y(f_i)$ and the reactive couple M_i will be replaced by $-H_i y'(f_i)$, where K_i and H_i are the deflection and rotation spring constants, respectively. Again, two additional equations are obtained from (10) for each cantilever support, so that there are always enough equations to obtain the values of the unknown reactions.

If the beam is supported solely by the Winkler foundation, then (10) gives the complete expression for y(x) in terms of the known applied forces and couples since $R_i = V_i = Z_i = 0$. Equation (10) is then equivalent to the result obtained by Miranda and Nair [2]. Note that the unknown M(0) and V(0) appear in [2] whereas these are absent in (10) because of the positioning of the origin in this paper. If the beam is additionally restrained at the ends only (single span beams), (10) is equivalent to the result obtained by Ting [3].

Another special case covered by (10) is that of a beam with simple and/or cantilever supports that are elastic but the Winkler foundation is absent. This is the situation encountered in a floor framework where beams rest on several perpendicular elastic beams. The value of k, the foundation modulus, will be set equal to zero in (10). When $k = 0$, many of the terms in (10) will be indeterminate of the form 0/0 so that L'Hospitals' rule must be used in the determination of deflections.

Results from using (10) were tested against those obtained by Hetenyi for several cases of loading and were found to be exactly the same for problems where Hetenyi obtained exact solutions. For problems where Hetenyi obtained approximate solutions, the results obtained by Hetenyi were found to be close to the exact solution given by (10). One of the examples given by Ting [3] is the beam in Figure 2. A comparison of the deflections obtained by Ting with those using (10) is shown in Table 2.



$k = 3 \times 10^6$ psf $\quad E = 519 \times 10^6$ psf $\quad I = 1.667$ ft^4

$K_1 = K_2 = 1.378 \times 10^6$ lb/ft^2 $\quad H_1 = H_2 = 11.019 \times 10^6$ ft-lb/rad

Figure 2

Table 2

x	y		y'		M		V	
ft	in		radians		ft-kips		kips	
	Ting	Eq.10	Ting	Eq.10	Ting	Eq.10	Ting	Eq.10
0+	.0000	-.0002	.0000	-.0000	.12	.12	-.03	.024
6	.0000	-.0006	.0000	-.0000	-.09	.003	-.24	-.24
12	-.0012	-.0011	.0000	.0000	-5.14	-5.11	-1.59	-.1.6
18	.0012	.0013	.0001	.0001	-18.07	-18.07	-2.15	-2.15
24	.012	.012	.0002	.0002	-12.78	-12.79	6.62	6.62
30	.025	.025	.0000	.0000	104.91	104.91	36	36

It will be noted from the Table 2 above that, while the two sets of values agree in the middle third of the beam, there are appreciable differences near the ends of the beam. These differences are again present in the next example for which values obtained by Hetenyi, Ting, and this writer are compared.

The second example considered by Ting is the same beam in Figure 2 except that the end restraints are rigid simple supports. To simulate a rigid support, Ting used $H_1 = H_2 = 0$ and a very large spring constant ($K_1 = K_2 = 90$ gigapounds per ft). Hetenyi found the solution for this problem to be

$$y(x) = \frac{P\beta\left[\cos\beta(L/2-x)\sinh\beta(L/2+x) - \cosh\beta(L/2-x)\sin\beta(L/2+x)\right.}{2k[\cosh\beta L + \cos\beta L]}$$

$$y'(x) = \frac{-P\beta^2\left[\sinh\beta(L/2-x)\sin\beta(L/2+x) + \sinh\beta(L/2-x)\sinh\beta(L/2+x)\right.}{2k[\cosh\beta L + \cos\beta L]}$$

$$M(x) = \frac{P\left[\cosh\beta(L/2-x)\sinh(L/2+x) - \sinh\beta(L/2-x)\cos\beta(L/2+x)\right.}{2k[\cosh\beta L + \cos\beta L]}$$

$$+\frac{P\left[\cos\beta(L/2-x)\sinh\beta(L/2+x) - \sin\beta(L/2-x)\cosh\beta(L/2+x)\right.}{4k[\cosh\beta L + \cos\beta L]}$$

$$V(x) = \frac{P\left[\cosh\beta(L/2-x)\cos\beta(L/2+x) + \cos\beta(L/2-x)\cosh\beta(L/2+x)\right.}{2k[\cosh\beta L + \cos\beta L]}$$

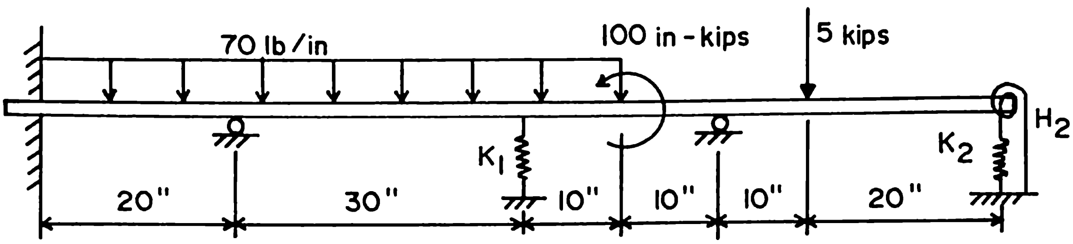
Table 3
Values of Deflection, Slope, Bending Moment, and Shear for Beam in Figure 2
(Elastic Restraints Replaced by Rigid Simple Supports)

x	Source	y	y'	M	V
ft		in	radians	ft-kips	kips
	Hetenyi	0.00000	0.00001	0.00000	-0.17644
0+	Ting	0	0	0	0.18
	Eq.10	0.00000	0.00001	0.00000	-0.17644
	Hetenyi	-0.00060	0.00001	0.21866	0.25384
6	Ting	-0.0012	0	.22	-0.25
	Eq.10	-0.00060	0.00001	0.21866	0.25384

	Hetenyi	-0.00109	0.00000	-5.01290	1.62222
12	Ting	-0.0012	0	-5.01	-1.62
	Eq.10	-0.00109	0.00000	-5.01290	1.62222
	Hetenyi	0.00130	-0.00008	-18.06307	2.15792
18	Ting	0.0012	0.0001	-18.06	-2.16
	Eq.10	0.00130	-0.00008	-18.06307	2.15792
	Hetenyi	0.01211	-0.00022	-12.79707	-6.62217
24	Ting	0.012	0.0002	-12.80	6.62
	Eq.10	0.01211	-0.00022	-12.79707	-6.62217
	Hetenyi	0.02471	0.00000	104.90600	-36.00000
30	Ting	0.025	0	104.90	36.00
	Eq.10	0.02471	0.00000	104.90600	-36.0000

Table 3 compares the results obtained by Hetenyi, Ting, and this writer. It will be noted that the results obtained by Hetenyi are exactly the same as the results obtained from Eq.10 of this paper.

The last example in this paper treats a beam (Figure 3) which contains all the types of loads and supports appearing in Eq. 10. The values of deflection, slope, bending moment, and shear for this beam appear in Table 4.



$k = 100$ lb/in $\quad E = 2 \times 10^6$ psi $\quad I = 32$ in^4

$k_1 = 5 \times 10^4$ lb/in $\quad K_2 = 10^5$ psi $\quad H_2 = 15 \times 10^4$ in -lb/rad

Figure 3

Table 4

x	y	y'	M	V
in	in	radians	in-kips	kips
0+	0.0000	0.0000	4.14	-0.26
10	-0.0021	-0.0003	-1.97	-0.96
20	-0.0002	0.0010	-15.10	-1.66
30	0.0169	0.0020	0.41	1.21
40	0.0343	0.0012	9.09	0.53
50	0.0384	-0.0004	11.10	-0.13
60	0.0157	-0.0050	45.82	3.14
70	0.0000	0.0010	-22.80	3.14
80	0.0175	0.0014	17.90	4.08
90	0.0201	-0.0007	8.75	-0.90
100	0.0089	-0.0013	-0.20	-0.89

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Notation

The following symbols are used in this paper:

a_i, b_i = x coordinates of distributed load of intensities A_i and B_i, respectively;

A, C_i, D_i, F_i, G, H, J, N_i, Q_i, S_i, T, U = coefficients;
c_i = x coordinate of point of application of force P_i;
d_i = x coordinate of point of application of couple Z_i;
E = modulus of elasticity of the beam;
f_i = x coordinate of cantilever i;
g_i = x coordinate of simple support;
H_i = rotation spring constant of yielding cantilever at f_i;
I = moment of inertia of the beam;
k = foundation modulus;
k_i = deflection spring constant of yielding simple support at g_i
Ki = deflection spring constant of yielding cantilever at f_i;
L = length of the beam;
M_i = reactive couple at cantilever located at f_i;
M(x) = bending moment at x;
q(x) = load function at x;
R_i = reaction at simple support located at g_i;
u = unit step function;
V_i = reactive force at cantilever located at g_i;
V(x) = shear force at x;
y(x) = deflection at x;
y'(x) = slope at x;
Z_i = couple applied at d_i;
δ = Dirac delta function;
δ' = unit doublet;
Φ_0, Φ_1, Φ_2, Φ_3 = Miranda-Nair functions.