STOCHASTIC MODELING AND FORECASTING OF RESERVOIR INFLOWS: A CASE STUDY IN THE PHILIPPINES*

by

Leonardo Q. Liongson**

ABSTRACT

This paper reports the general findings of a research project for stochastic modeling and forecasting of monthly and five-day inflows to the multi-purpose Angat reservoir. The stochastic models investigated were the pure-runoff autoregressive-moving-average (ARMA) types, both of seasonal and nonseasonal forms, and either with or without state estimation techniques or Kalman filtering; and rainfall-runoff ARMA-type or ARMAX (ARMA with exogenous input). The research demonstrated the applicability of the best-selected models for forecasting dry-season low flows and wet-season moderate flows. Recommendations for possible model improvements were also made.

INTRODUCTION

The government of the Philippines has implemented an intensive countrywide infrastructure program in water resources for the past two decades. This has been aimed at maximizing the benefits that the nation as a whole can derive from its increasingly scarce water resource for its rapidly growing population. Among these projects are multipurpose dams, hydroelectric power schemes, flood control and warning systems, as well as irrigation and water supply distribution systems. Proper management of these projects could, among other benefits, boost the economy by way of increased agricultural and energy production, and raise the standard of living through the provision of safe and adequate water supply and the mitigation of flood hazards and damages.

*Paper presented at the **UNESCO/ROSTSEA Regional Workshop on Streamflow Forecasting**, National Hydraulic Research Center/College of Engineering, University of the Philippines, 17-22 October 1988.

**Professor, Department of Engineering and Computer Sciences and Research Fellow, National Hydraulic Research Center, University of the Philippines

The benefits accruing to water resources development projects are naturally realized only during operations. Water, the basic resource, is recognized as a renewable resource, yet its occurence is complicated by uncertainties with regard to time, place, quantity, and quality. The hydrological cycle, although understood to be basically controlled by the astronomical and climatological cycles which bring about the seasons, contains aspects of multi-scaled randomness or uncertainty associated with highly variable atmospheric and basin conditions. These uncertainties perennially pose serious challenge and problems to managers and operators of water resource systems. Proper, reliable, responsive, and safe operations would therefore require a dependable and accurate scheme for forecasting the occurence and quantity of water inflows. Past historical hydrological data and criteria derived therefrom, originally utilized for design, are no longer sufficient bases for operations.

Stochastic or time-series models afford easily portable and practical real-time hydrological forecasting methods. These need not be complex mathematical models and software systems; on the contrary, microcomputer systems can easily accomodate the speed and memory requirements of the said models. To obtain fast response and adaptive functioning of these models in real time, model parsimony (or simplicity) and flexibility are always aimed at. Properly trained junior engineers can easily operate these computer systems in conjunction with hydrometric data updates communicated from the field. When more powerful computer resources become available to forecasting offices and extensive data collection schemes are put in place, the use of more sophisticated and data-extensive physically-based models and systems is nevertheless encouraged.

DEVELOPMENT OF FORECASTING MODELS

In the years 1985 to 1987, the National Hyraulic Research Center (NHRC) in cooperation with the Department of Engineering and Computer Sciences of the U. P. College of Engineering undertook a research and development activity for developing and evaluating stochastic streamflow forecasting models (Liongson *et al*, 1988). The objective was to develop a class of hydrological forecasting models for water inflows, suitable for applications to the real-time operations of multipurpose dam and storage facilities, thereby potentially serving the operational requirements for irrigation, hydropower, flood control, municipal water supply and other purposes. In view of the importance of the Angat reservoir (Fig. 1) as the major source of water supply for Metro-Manila, hydro-electric energy for the Luzon power grid, and irrigation water for the farmlands in provinces north of Manila, water inflows to this site was selected as the subject of a case study. Available post-World War II streamflow, rainfall, and Angat reservoir operations data were gathered, screened, and utilized for modeling purposes. The data series was divided into two roughly equal segments: pre-construction data for model building and calibration, and post-construction data for model verification.

Several structurally distinct types of time-series models were investigated. As a class of models, these models have been extensively studied, applied, and documented in both hydrological and statistical professional literature (Box and Jenkins, 1976; Bras and Rodriguez-Iturbe, 1985; Salas *et al*, 1980). In this study, the particular types considered were:

(a.) Pure runoff ARMA (autoregressive-moving-average) models, modified Box-Jenkins type, both nonseasonal and seasonal forms (Hipel *et al*, 1977; McCleod *et al*, 1977; Salas *et al*, 1982; Salas and Obeysekera, 1982; Tao and Delleur, 1976).

(b.) Pure runoff ARMA models, with state-estimation techniques or Kalman filtering, which is based on the theory of minimizing the forecast error variance given a model forecast and a subsequent measurement (O'Connel, 1980).

(c.) Rainfall-runoff ARMA-type models or ARMAX (ARMA with exogenous input), with state estimation techniques or Kalman filtering (O'Connel, 1980).

In addition, two alternative scales of forecast lead time and time unit of prediction were selected, on the basis that potential model application would be limited to medium-term real-time Angat operations for hydropower, water supply, and irrigation:

(a.) Aggregate 5-days

(b.) Monthly

The methods of model parameter estimation were:

(a.) Off-line or "block-data" estimation method, resulting in flow forecasts with fixed parameters.

(b.) On-line recursive estimation method, resulting in combined flow and variable parameter estimates.

Listings of generic ARMA and Kalman filtering programs for the mainframe computer were secured from research contacts abroad and were modified, adapted, and augmented for implementation with the PC-compatibles at NHRC.

MODEL TYPES AND FEATURES

Four (4) distinct models were developed and these are described as follows:

(a.) Monthly Runoff nonseasonal ARMA(p,q); fixed parameters
(Note: seasonality is considered in the mean and variance but not in the ARMA parameters)

$$Z_t = \Phi_1 Z_{t-1} + \Phi_2 Z_{t-2} + \ldots + \Phi_p Z_{t-p} + \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2} - \ldots - \theta_q \epsilon_{t-q}$$

where t = monthly time subscript

Z_t = transformed and seasonally standardized flow deviate:

$$= \frac{F(Q_t) \text{ - seasonal mean of } F(Q_t)}{\text{seasonal std. deviation of } F(Q_t)}$$

where F(.) is either the identity, power, or logarithmic transformation, required to skew-normalize the original flow data Q_t. This transformation applies likewise to the other models described below.

p, q = orders of the autoregressive and moving average terms, respectively

$\Phi_1, \Phi_2, \ldots \Phi_p$ = autoregressive parameters

$\theta_1, \theta_2, \ldots \theta_q$ = moving-average parameters

ε_t = independent random error or residual

When q = 0, the model becomes simply an AR(p) or autoregressive model:

$$Z_t = \Phi_1 Z_{t-1} + \Phi_2 Z_{t-2} + \ldots + \Phi_p Z_{t-p} + \epsilon_t$$

On the other hand, when p = 0, an MA(q) or moving-average model is obtained:

$$Z_t = \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2} - \ldots - \theta_q \epsilon_{t-q}.$$

(b.) 5-Days or Monthly Runoff AR(p) with Kalman Filtering

(1.) Pure State Estimation (fixed parameters)

System Equation:

$$\begin{bmatrix} Y_t \\ \cdot \\ \cdot \\ \cdot \\ Y_{t-p+1} \end{bmatrix} = \left[\begin{array}{ccc:c} \Phi_1 & \cdots & \Phi_{p-1} & \Phi_p \\ \hline & I_{p-1} & & 0 \end{array}\right] \begin{bmatrix} Y_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ Y_{t-p} \end{bmatrix} + \begin{bmatrix} W_t \\ 0 \end{bmatrix}$$

Measurement Equation:

$$Z_t = [\ 1\ \ 0\ \ \ldots\ \ 0\ ] \begin{bmatrix} Y_t \\ \cdot \\ \cdot \\ \cdot \\ Y_{t-p+1} \end{bmatrix} + V_t$$

where t = 5-days or monthly time subscript

Y_t = model flow deviate (transformed and seasonally standardized)

Z_t = measured flow deviate

$\Phi_1, \ \ldots, \ \Phi_{p-1}, \ \Phi_p$ = AR parameters (fixed)

W_t = model noise

V_t = measurement noise

I_{p-1} = (p-1) x (p-1) identity matrix

(2.) Combined State-Parameter Estimation (variable parameters)

System Equation:

$$
\left[\begin{array}{c} Y_t \\ \vdots \\ Y_{t-p+1} \\ \hline \Phi_{1t} \\ \vdots \\ \Phi_{pt} \end{array}\right]
=
\left[\begin{array}{cc:c} \Phi_{1t} \ldots \Phi_{p-1t} & \Phi_{pt} & 0 \\ I_{p-1} & 0 & 0 \\ \hline 0 & 0 & I_p \end{array}\right]
\left[\begin{array}{c} Y_{t-1} \\ \vdots \\ Y_{t-p} \\ \hline \Phi_{1t-1} \\ \vdots \\ \Phi_{pt-1} \end{array}\right]
+
\left[\begin{array}{c} W_t \\ 0 \\ \hline 0 \end{array}\right]
$$

Measurement Equation:

$$
\left[\begin{array}{c} Z_t \\ Z_t \end{array}\right]
=
\left[\begin{array}{cccc:ccc} 1 & 0 & \ldots & 0 & & 0 & \\ & & 0 & & Y_{t-1} & \ldots & Y_{t-p} \end{array}\right]
\left[\begin{array}{c} Y_t \\ \vdots \\ Y_{t-p+1} \\ \hline \Phi_{1t} \\ \vdots \\ \Phi_{pt} \end{array}\right]
+
\left[\begin{array}{c} V_{t1} \\ V_{t2} \end{array}\right]
$$

where t = 5-days or monthly time subscript

Y_t = model flow deviate

$\Phi_{1t}, \ldots, \Phi_{pt}$ = AR parameters (variable)

Z_t = measured flow deviate

W_t = model noise

V_{1t}, V_{2t} = measurement noise

I_p = (p x p) identity matrix

I_{p-1} = (p-1) x (p-1) identity matrix (c.) 5-Days or Monthly Rainfall-Runoff ARMAX(p,0,1) with Kalman Filtering

(1.) Pure State Estimation (fixed parameters)

System Equation:

$$\begin{bmatrix} Y_t \\ \\ P_t \end{bmatrix} = \left[\begin{array}{cc:c} \multicolumn{2}{c:}{\Phi^T} & \delta \\ I_{p-1} & 0 & 0 \\ 0 & 0 & \beta \end{array}\right] \begin{bmatrix} Y_{t-1} \\ \\ P_{t-1} \end{bmatrix} + \begin{bmatrix} W_{1t} \\ \\ W_{2t} \end{bmatrix}$$

Measurement Equation:

$$\begin{bmatrix} Z_t \\ R_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} Y_t \\ P_t \end{bmatrix} + \begin{bmatrix} V_{1t} \\ V_{2t} \end{bmatrix}$$

where t = 5-days or monthly time subscript

Y_t = (p x 1) model flow deviate vector

P_t = model rainfall deviate

ϕ = (p x 1) AR parameter vector

δ = rainfall-runoff coefficient

β = rainfall autoregressive coefficient

Z_t = measured flow deviate

R_t = measured rainfall deviate

W_{1t}, W_{2t} = model noise

V_{1t}, V_{2t} = measurement noise

(2.) Combined State-Parameter Estimation (variable parameters)

System Equation:

$$
\begin{bmatrix} Y_t \\ P_t \\ \hline \Phi_t \\ \delta_t \\ \beta_t \end{bmatrix}
=
\left[\begin{array}{cccc:c}
\Phi_{t-1}^T & & \delta_{t-1} & & 0 \\
I_{p-1} & 0 & 0 & & 0 \\
0 & 0 & \beta_{t-1} & & 0 \\
\hline
0 & 0 & 0 & & \\
0 & 0 & 0 & & I_{p+2} \\
0 & 0 & 0 & &
\end{array}\right]
\begin{bmatrix} Y_{t-1} \\ P_{t-1} \\ \hline \Phi_{t-1} \\ \delta_{t-1} \\ \beta_{t-1} \end{bmatrix}
+
\begin{bmatrix} W_{1t} \\ W_{2t} \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}
$$

Measurement Equation:

$$
\begin{bmatrix} Z_t \\ R_t \\ \hline Z_t \\ R_t \end{bmatrix}
=
\left[\begin{array}{c:ccc}
1\ldots0 & 0 & 0 & 0 \\
0\ldots1 & 0 & 0 & 0 \\
\hline
0\ldots0 & Y_{t-1}^T & P_{t-1} & 0 \\
0\ldots0 & 0 & 0 & P_{t-1}
\end{array}\right]
\begin{bmatrix} Y_t \\ P_t \\ \hline \Phi_t \\ \delta_t \\ \beta_t \end{bmatrix}
+
\begin{bmatrix} V_{1t} \\ V_{2t} \\ \hline V_{3t} \\ V_{4t} \end{bmatrix}
$$

where ϕ_t = (p x 1) AR parameter vector (variable)

δ_t = variable rainfall-runoff coefficient

β_t = variable rainfall autoregressive coefficient,

and the other variables are similarly defined as before.

(d.) Seasonal ARMA Models - Pure Runoff Models with Seasonally Varying ARMA Parameters.

$$
Y_{10+v,t} = \sum_{k=1}^{p(v)} \Phi_k(v)\, Y_{10+v,t-k} + \epsilon_{10+v,t} - \sum_{k=1}^{q(v)} \Theta_k(v)\, \epsilon_{10+v,t-k}
$$

where i = year index

s = number of seasons (12 months)

v = season to season index = 1, 2, 3, ... s

$t, t\text{-}k$ = within season time indeces

= 1,2,...6 five-day groups

$Y_{is+v,t}$ = flow deviate for year i, season v, 5-day t

$p(v)$ = AR order for season v

$\phi_k(v)$ = lag-k AR parameter for season v

$q(v)$ = MA order for season v

$\theta_k(v)$ = lag-k MA parameter for season v

$\varepsilon_{is+v,t}$ = independent error or residual

Modeling and Forecasting Results

For the Angat case study, the modeling effort was divided into four phases:

(a.) ARMA Modeling and Forecasting of Monthly Streamflows

A step-by-step procedure (Figs. 2 and 3) of ARMA model construction - identification, estimation, and diagnostics - was used to model and forecast the monthly inflows to Angat reservoir. Statistical properties (mean, standard deviation, correlation coefficient, standard forecast error, and peak forecast error) on the one-step ahead (one-month) forecast for the various competing ARMA models were computed in order to compare the actual performance and to check if the forecast preserved the statistical properties of the historical data. The AR(3) model comes out best, followed by AR(1) as second best (Figs. 4 and 5).

(b.) ARMA Modeling and Forecasting with Kalman Filtering of Monthly Streamflows

The use of state estimation techniques or Kalman filtering (Fig. 6) for monthly streamflow forecasting was also investigated. Time series models of the ARMA and ARMAX types were cast within the state-space framework of the Kalman filter (Figs. 7, 8, and 9) and used to forecast the one-month ahead inflows at the Angat reservoir. The forecasts were of two types, pure state estimation and combined state-parameter estimation. The former assumes that the model parameters were time invariant while the converse was assumed in the latter. The performance of the identified forecasting models were compared and then evaluated through the use of the mean, standard deviation, correlation coefficient, standard forecast error, and peak forecast error of the forecasted series, relative to those of the historical series. The model selected to be used within the Kalman filter framework was a combination of the AR(1) and AR(3) models (Figs. 10, 11, 12, 13, and 14).

(c.) Seasonal ARMA Modelling and Forecasting of 5-Day Streamflows

A seasonal forecasting model (Figs. 15 and 16) for the 5-day inflows of the Angat reservoir was developed using the ARMA modeling methodology for stationary time series. In developing the model, three assumptions were made, namely: (1) the periodicity of the time series is monthly whereof the number of seasons in a year is twelve; (2) each month is composed of six average five-day inflows; and (3) the parameters of the model of each month can be obtained independently from the other months. The best fit models for the independent monthly series were arrived at by following the ARMA modeling procedure and the parameter estimates of the models were taken as the periodic parameters of the seasonal forecasting model. The forecasting model was evaluated by comparing the generated lead one forecasted inflows to the observed data in terms of the mean, standard deviation, correlation coefficient, standard forecast error, and peak forecast error. The forecasts were found to closely approximate the measured low inflows and the recession limbs of moderate floods. On the other hand, the forecasts generally underestimated the observed peak inflows and the recession limbs of high floods. The twelve seasonal models were:

January -	MA(1)	July -	AR(1)
February -	ARMA(1,2)	August -	MA(2)
March -	MA(1)	September -	MA(1)
April -	MA(2)	October -	MA(1)
May -	MA(1)	November -	AR(1)
June -	AR(2)	December -	MA(3)

(d.) ARMA Modeling and Forecasting with Kalman Filtering of 5 - Day Streamflows

State-space techniques and Kalman filtering were applied to the class of ARMA models for the purpose of real-time forecasting of 5-day inflows to the Angat reservoir. Unknown noise statistics were estimated using an adaptive recursive estimation algorithm. Forecasts were obtained using both state estimation and combined state-parameter estimation techniques. Combined state-parameter estimation allowed the model parameters to be recursively updated (Fig. 17). The use of the multivariate ARMAX model to take explicitly into account the effect of rainfall on the runoff process was also considered. Model performance was evaluated by comparing important statistical properties of the observed and forecasted series - mean, standard deviation, correlation coefficient, standard forecast error, and peak forecast error. The AR(1) model was considered best, with only minor improvement with the ARMAX model (Figs. 18 and 19).

GENERAL CONCLUSIONS AND RECOMMENDATIONS

Under the different model options and comparison criteria, it was hard to discriminate in a straightforward manner among the few but best competing models of the ARMA type. They all provided the forecasts close to the measured values of dry season low flows and recession limbs of wet season moderate flows. On the other hand, they all underestimated peak flows as well as recession limbs of extremely high floods. The minor differences in the statistical performances of competing models occured only in small and sometimes insignificant degrees. In any case, the derived models were judged to be suitable for forecasting dry season flows and wet season moderate flows.

The above results were not surprising and in fact could be anticipated since ARMA models or their variations esentially exploit and parameterize the short memory property of hydrologic processes, best exemplified by flow recession. The reservoir inflow models, confined to the basic ARMA type, were not, per se, flood forecasting models. The time scales adopted, namely, one month and five days, were too long for possible accurate forecasts to be made of high flows derived from short duration random storm rainfall events. Flood forecasting models, not necessarily of the ARMA type, which could better perform should contain physically based model variables such as short-duration rainfall forecasts, basin loss and effective rainfall amounts, and basin and channel hydrograph routing parameters. The latter would also deal with shorter time scales (as lead time and time unit of forecasts) ranging from fractions of an hour to at most two or three days.

The state estimation techniques or Kalman filtering applied to the ARMA models did not contribute any major improvement in the forecasting performance relative to the modified Box-Jenkins ARMA modeling and forecasting procedure. Nevertheless, the adoption of state-estimation techniques was recommended since it allowed in general for making combined flow forecasts and recursive parameter updates as more flow measurements became available. On the other hand, seasonal 5-day ARMA models did not show any improvement over the nonseasonal 5-day ARMA models.

The monthly and 5-day forecasting models which used state-estimation techniques may be improved by incorporating additional variables (state or exogenous) such as antecedent basin moisture condition or soil moisture, basin losses, or else effective rainfall.

ACKNOWLEDGEMENTS

The author wishes to acknowledge the financial support provided by the U. P Engineering Research and Development Foundation, Inc. (UPERDFI) to the research project "Development of Hydrological Forecasting Models for Real-time Operations of Water Resource Systems" upon which this paper was based. Thanks and gratitude are also expressed to Director and Prof. A. A. Alejandrino of NHRC for the research cooperation and organizational support. The author is also greatly indebted to the three tireless junior researchers, Mssrs. J. M. B. Blanco, C. Q. Dizon, D. S. Rojas, and R. C. Villa, all NHRC research associates, who implemented the detailed modeling tasks, helping this project realize its objectives, while on their part getting credit for thesis work towards their M. S. (Water Resources) degrees. The hospitality and data support accorded by the National Power Corporation, Angat Dam Office, are also acknowledged.

REFERENCES

Box, G. E. P. and G. M. Jenkins (1976). Time Series Analysis: Forecasting and Control, Holden-Day, San Francisco, U. S. A.

Bras, R. L. and I. Rodriguez-Iturbe (1985). Random Functions and Hydrology, Addison-Wesley, Reading, Massachusetts.

Hipel K. W., *et al* (1977). Advances in Box-Jenkins Modeling: Model Construction, Water Resources Research, vol. 13, no. 3, pp. 567-576.

Liongson, L. Q., J. M. Blanco, C. Q. Dizon, D. S. Rojas, and R. C. Villa (1988). Final Report. Development of Hydrological Forecasting Models for Real-time Operations of Water Resources Systems, Vols. I-III, ERDFI-NHRC-UP. Quezon City.

McCleod, A. I., *et al* (1977). Advances in Box-Jenkins Modeling: Applications, Water Resources Research, vol. 13, no. 13, pp. 577-586.

O'Connel, P. E. (Ed.) (1980). Real-Time Hydrological Forecasting and Control, Proc. First International Workshop, Institutue of Hydrology, Wallingford, U. K.

Salas, J. D. (1980). Applied Modeling of Hydrologic Time Series, Water Resources Publications, Littleton, Colorado.

Salas, J. D., *et al* (1982). Estimation of ARMA Models with Seasonal Parameters, Water Resources Research, vol. 18, no. 4, pp. 1006-1010.

Salas, J. D. and J. T. B. Obeysekera (1982). ARMA Modeling of Hydrologic YTime series, Water Resources Research, vol. 18, no. 4, pp. 1011-1021.

Tao, P. C. and J. Delleur (1976). Seasonal and Nonseasonal ARMA Models in Hydrology, Journal of the Hydraulics Division, Proc. ASCE, vol. 10, no. HY10, pp. 1541-1559.

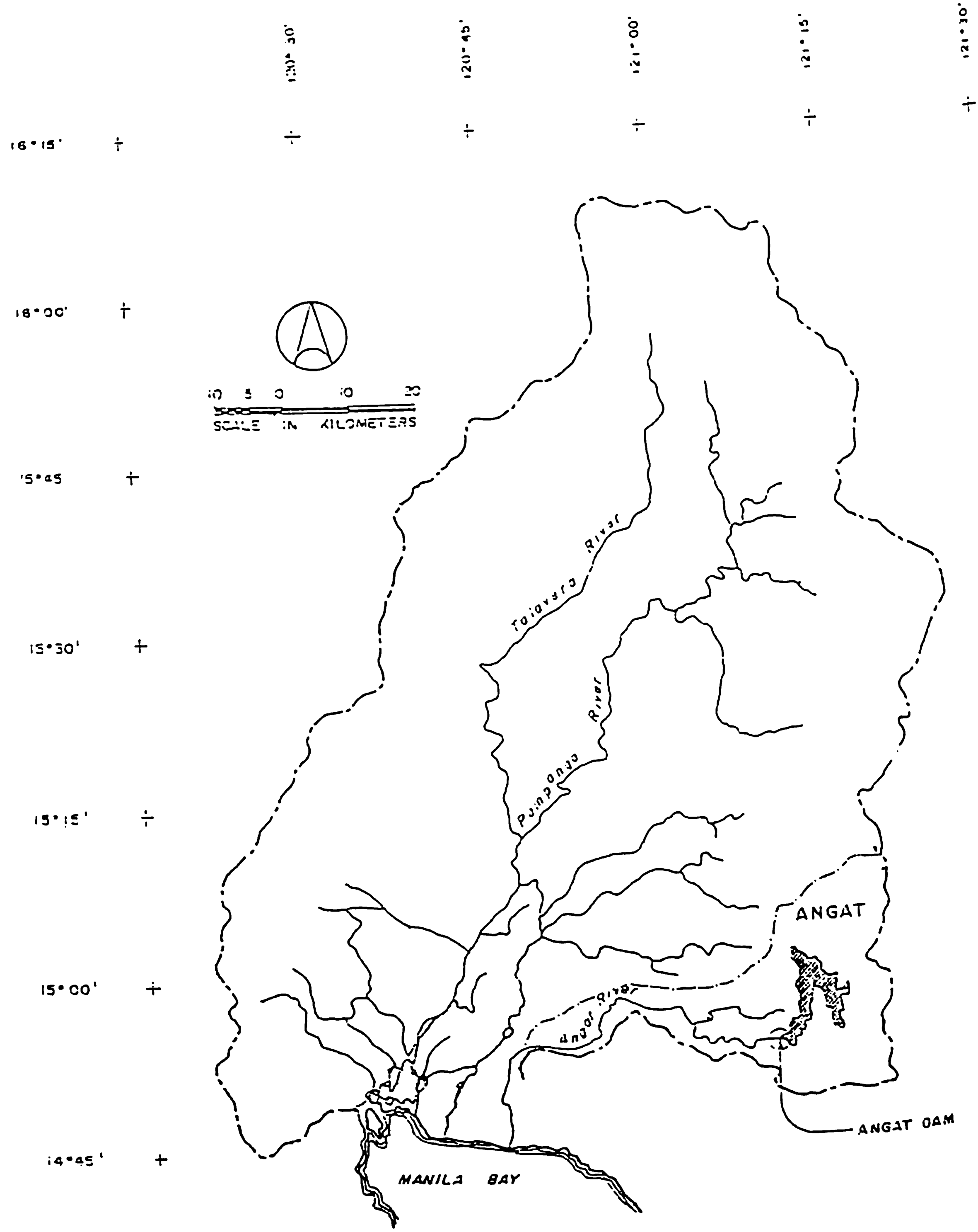


Figure 1. Pampanga River Watershed System, including Angat River and Dam.

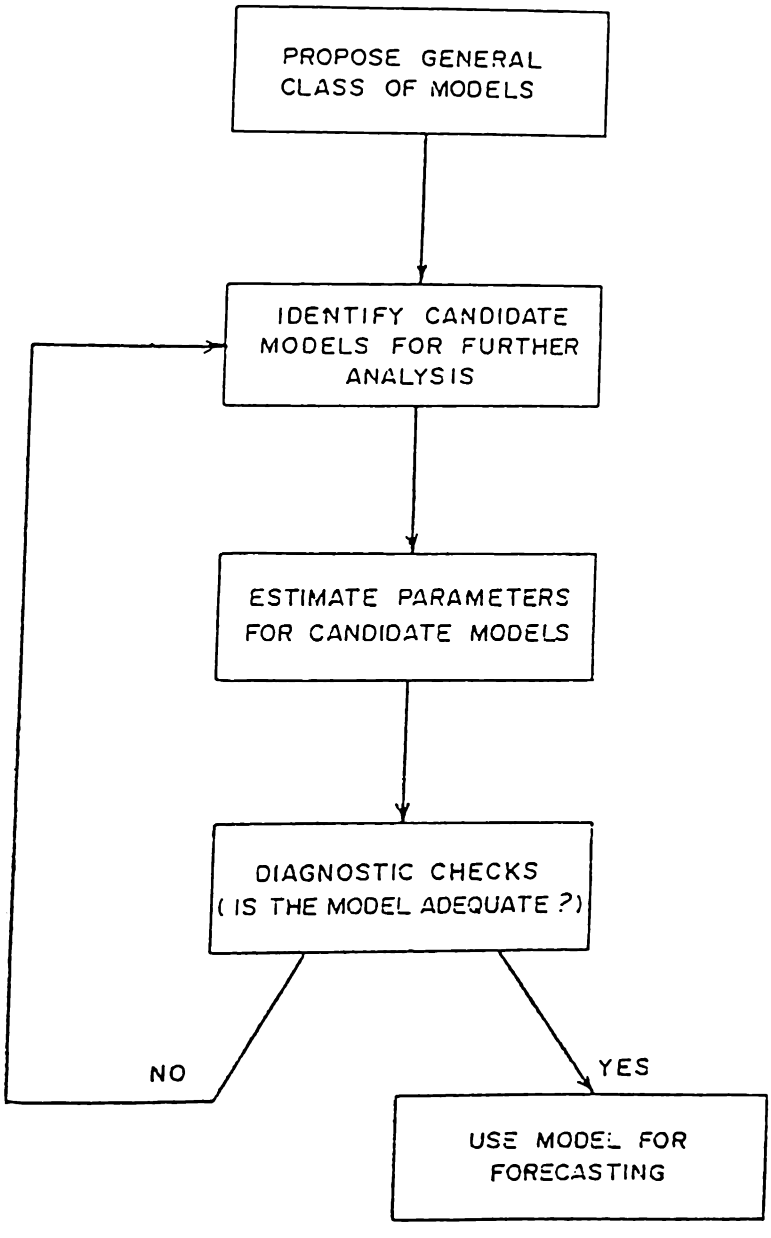


Figure 2. ARMA Modeling Procedure

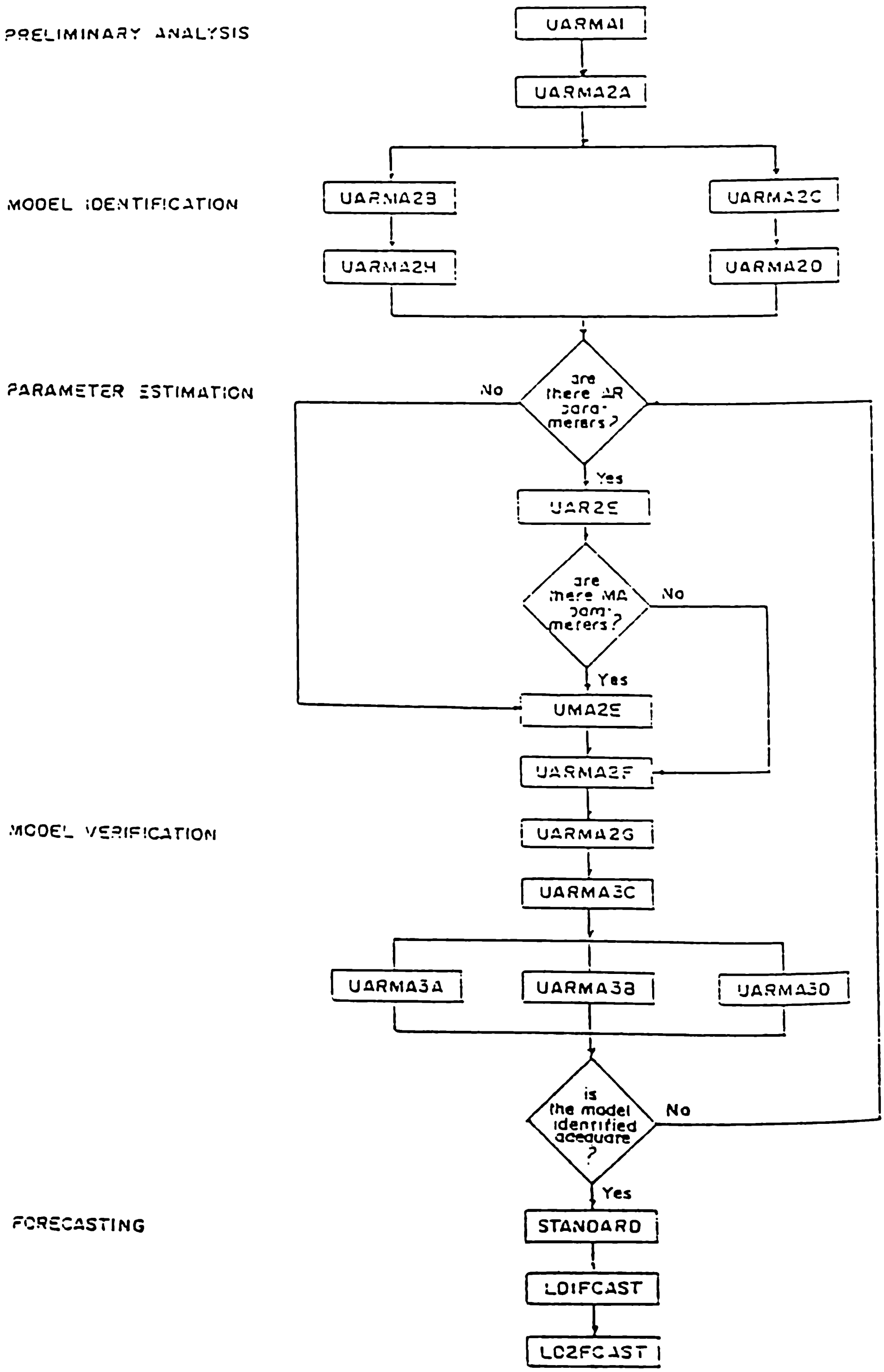
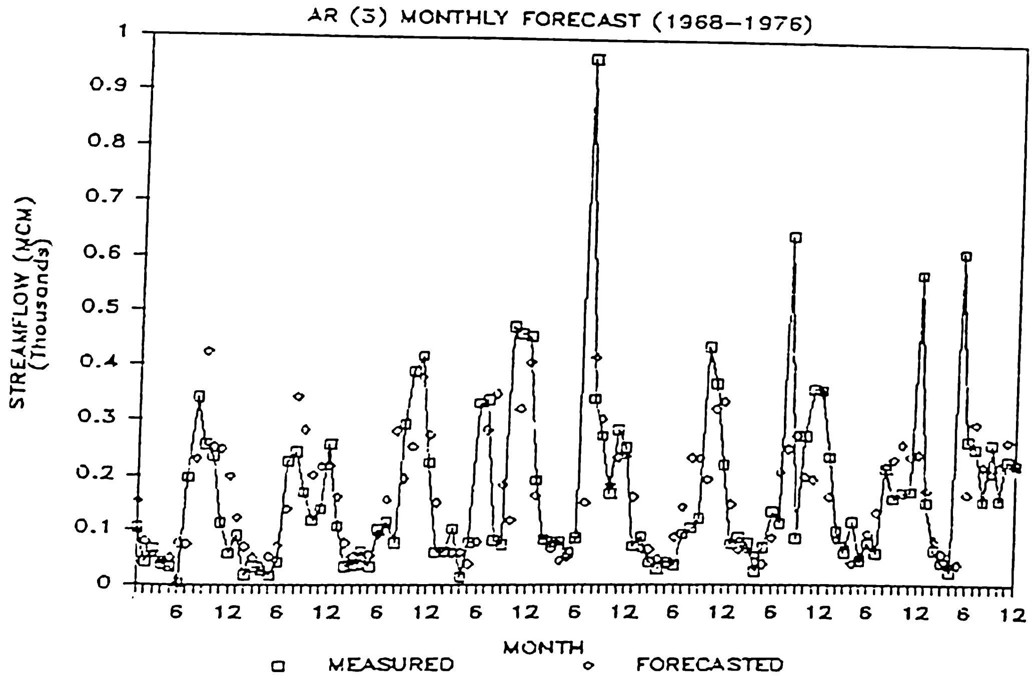
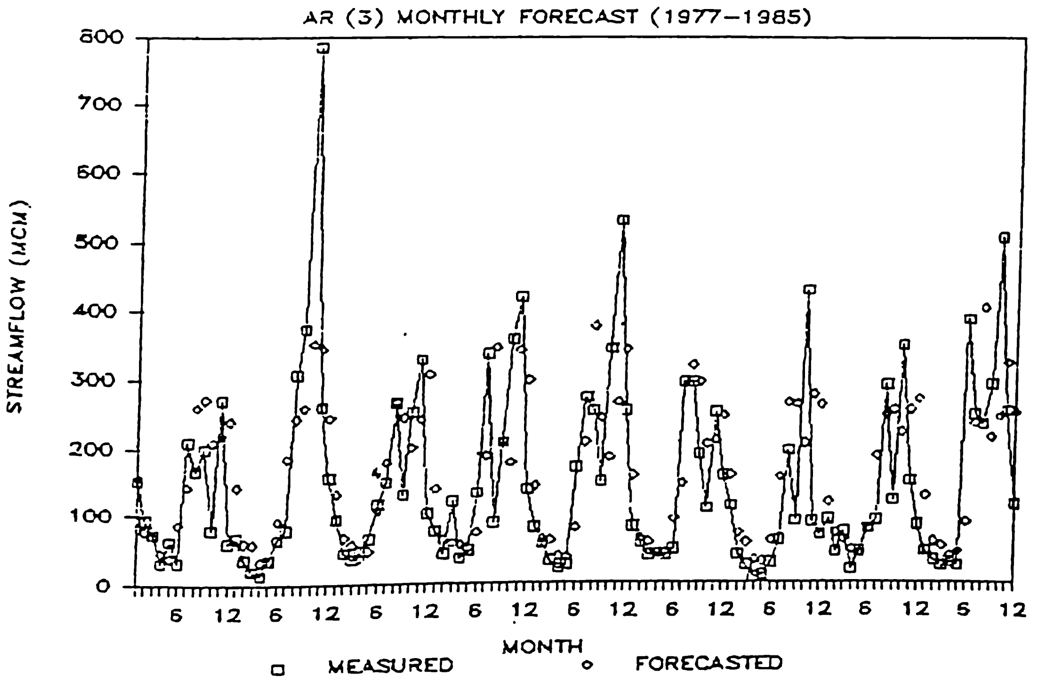


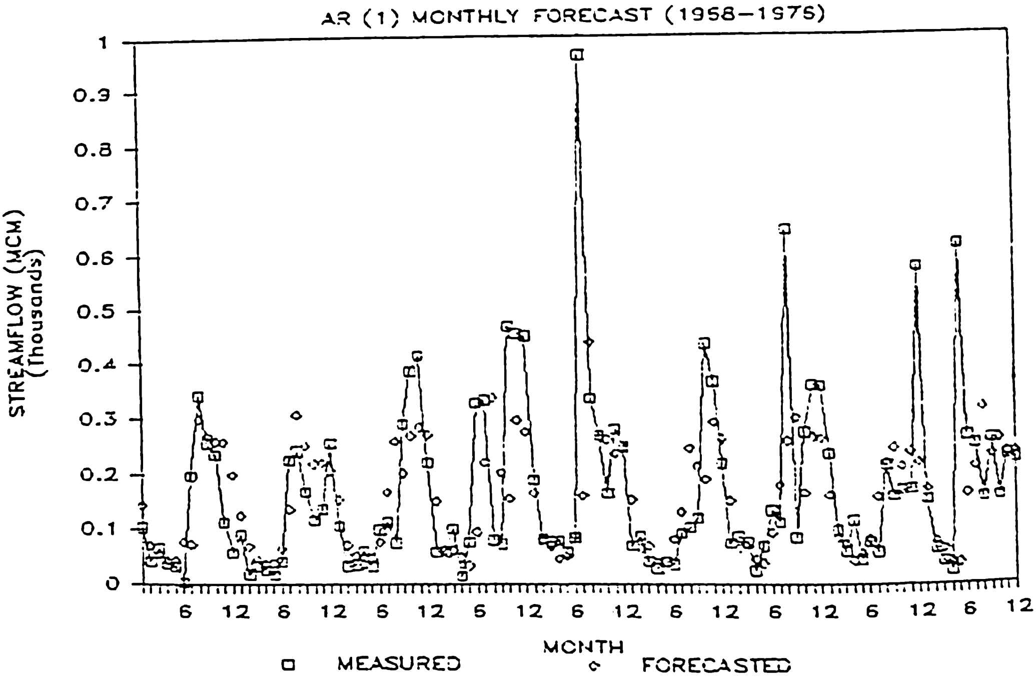
Figure 3. ARMA Modeling Flowchart

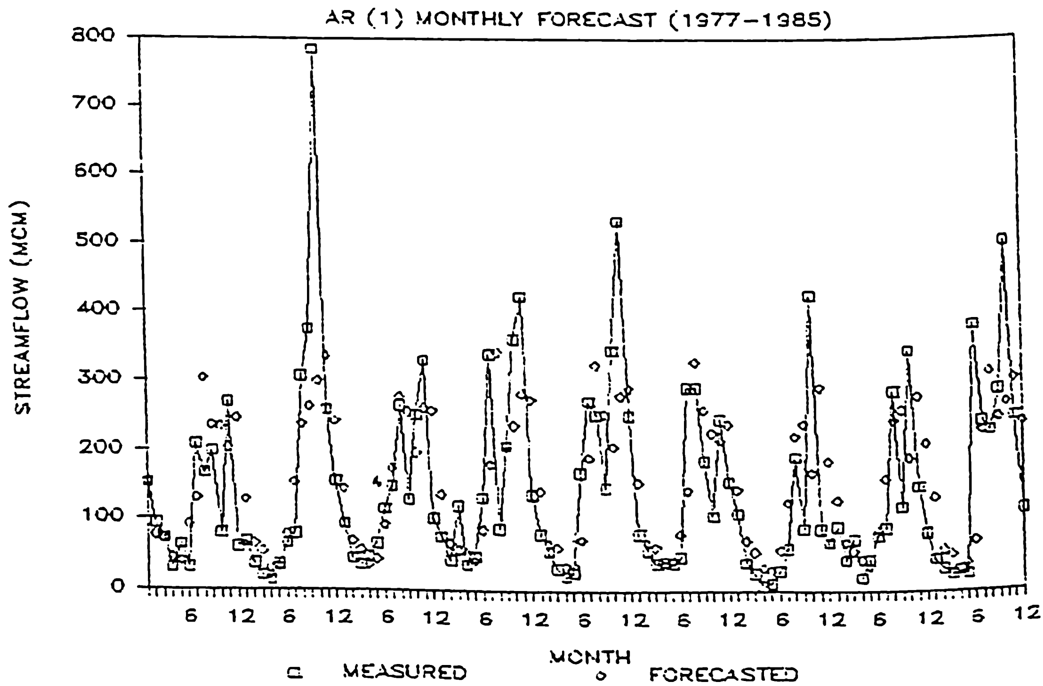




AR(3) Model Lead-1 Forecast (Validation Data)

Figure 4. AR (3) Monthly Forecast





AR (1) Model Lead-1 Forecast (Validation Data)

Figure 5. AR (1) Monthly Forecast

System Equation:

$$X_t = \Phi_{t-1}X_{t-1} + \Gamma_t W_t$$

Measurement Equation:

$$Z_t = H_t X_t + V_t$$

Propagated State:

$$\hat{X}_{t/t-1} = \Phi_{t-1}\hat{X}_{t-1/t-1} + \Gamma_t \bar{W}_t$$

Associated Error Variance-Covariance:

$$P_{t/t-1} = \Phi_{t-1}P_{t-1/t-1}\Phi_{t-1}^T + \Gamma_t Q_t \Gamma_t^T$$

Filter Innovation:

$$\nu_t = Z_t - H_t\hat{X}_{t/t-1} - \bar{V}_t$$

Filtered State:

$$\hat{X}_{t/t} = \hat{X}_{t/t-1} + K_t\nu_t$$

Kalman Gain:

$$K_t = P_{t/t-1}H_t^T(H_t P_{t/t-1}H_t^T + R_t)^{-1}$$

Associated Error-Variance Covariance:

$$P_{t/t} = (I - K_t H_t)P_{t/t-1}$$

Figure 6. Discrete Linear Kalman Filter

AR(1) Model

State Equation: $X_t = \Phi X_{t-1} + W_t$

$$\begin{bmatrix} Y_t \end{bmatrix} = \begin{bmatrix} \emptyset_{t-1} \end{bmatrix} \begin{bmatrix} Y_{t-1} \end{bmatrix} + \begin{bmatrix} W_t \end{bmatrix}$$

Measurement Equation: $Z_t = H_t X_t + V_t$

$$\begin{bmatrix} \overset{o}{Y}_t \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} Y_t \end{bmatrix} + \begin{bmatrix} V_t \end{bmatrix}$$

System and Measurement Noise Covariance Matrices:

$$Q = \begin{bmatrix} \sigma_{wt}^2 \end{bmatrix} \qquad R = \begin{bmatrix} \sigma_{vt}^2 \end{bmatrix}$$

AR(2) Model

State Equation: $X_t = \Phi X_{t-1} + W_t$

$$\begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix} = \begin{bmatrix} \emptyset_{1t-1} & \emptyset_{2t-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Y_{t-2} \end{bmatrix} + \begin{bmatrix} W_t \\ 0 \end{bmatrix}$$

Measurement Equation: $Z_t = H_t X_t + V_t$

$$\begin{bmatrix} \overset{o}{Y}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} V_t \end{bmatrix}$$

System and Measurement Noise Covariance Matrices:

$$Q = \begin{bmatrix} \sigma_{wt}^2 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \sigma_{vt}^2 \end{bmatrix}$$

Figure 7. State-space Formulation of AR (1) and AR (2) Models (State Estimation Problem)

AR(1) Model

State Equation: $X_t = \Phi X_{t-1} + W_t$

$$\begin{bmatrix} Y_t \\ \emptyset_t \end{bmatrix} = \begin{bmatrix} \emptyset_{t-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ \emptyset_{t-1} \end{bmatrix} + \begin{bmatrix} W_t \\ 0 \end{bmatrix}$$

Measurement Equation: $Z_t = H_t X_t + V_t$

$$\begin{bmatrix} Y^o_t \\ Y^o_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Y_{t-1} \end{bmatrix} \begin{bmatrix} Y_t \\ \emptyset_t \end{bmatrix} + \begin{bmatrix} V_{t1} \\ V_{t2} \end{bmatrix}$$

AR(2) Model

State Equation: $X_t = \Phi X_{t-1} + W_t$

$$\begin{bmatrix} Y_t \\ Y_{t-1} \\ \emptyset_{1t} \\ \emptyset_{2t} \end{bmatrix} = \begin{bmatrix} \emptyset_{1t-1} & \emptyset_{2t-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Y_{t-2} \\ \emptyset_{1t-1} \\ \emptyset_{2t-1} \end{bmatrix} + \begin{bmatrix} W_t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Measurement Equation: $Z_t = H_t X_t + V_t$

$$\begin{bmatrix} Y^o_t \\ Y^o_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & Y_{t-1} & Y_{t-2} \end{bmatrix} \begin{bmatrix} Y_t \\ Y_{t-1} \\ \emptyset_{1t-1} \\ \emptyset_{2t-1} \end{bmatrix} + \begin{bmatrix} V_{t1} \\ V_{t2} \end{bmatrix}$$

Figure 8. State-space Formulation of Combined State-Parameter Estimation Problem

ARMAX Model

State Equation: $X_t = \Phi X_{t-1} + W_t$

$$\begin{bmatrix} Y_t \\ P_t \end{bmatrix} = \begin{bmatrix} \emptyset_1 & \Omega_1 \\ 0 & \emptyset_{p1} \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ P_{t-1} \end{bmatrix} + \begin{bmatrix} w_{t1} \\ w_{t2} \end{bmatrix}$$

Measurement Equation: $Z_t = H_t X_t + V_t$

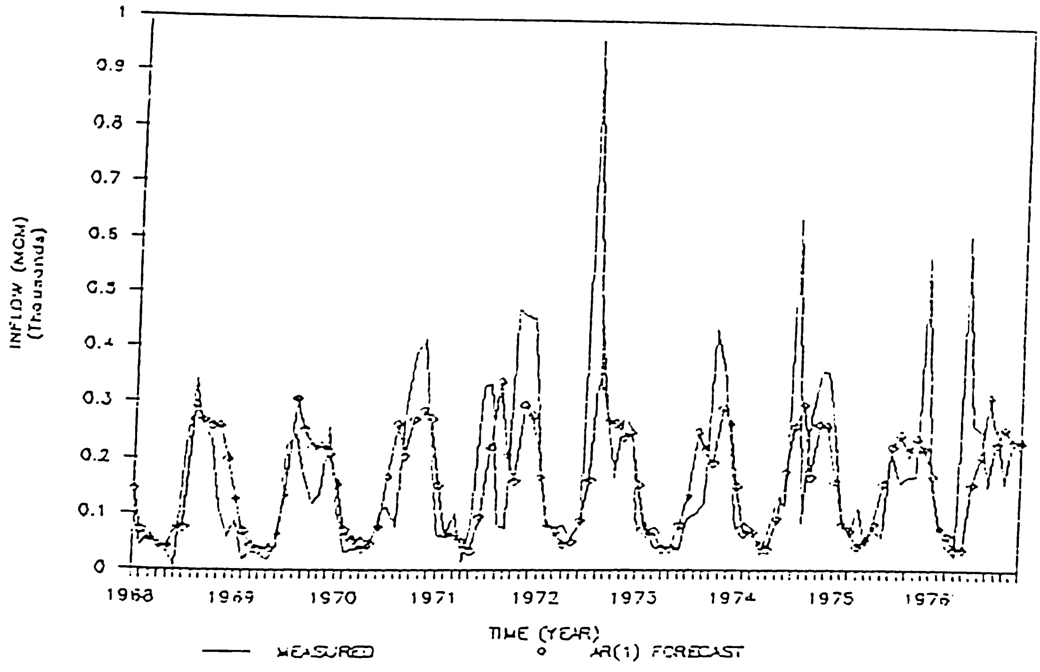
$$\begin{bmatrix} Y_t^o \\ P_t^o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_t \\ P_t \end{bmatrix} + \begin{bmatrix} v_{t1} \\ v_{t2} \end{bmatrix}$$

State and Measurement Noise Covariance Matrices:

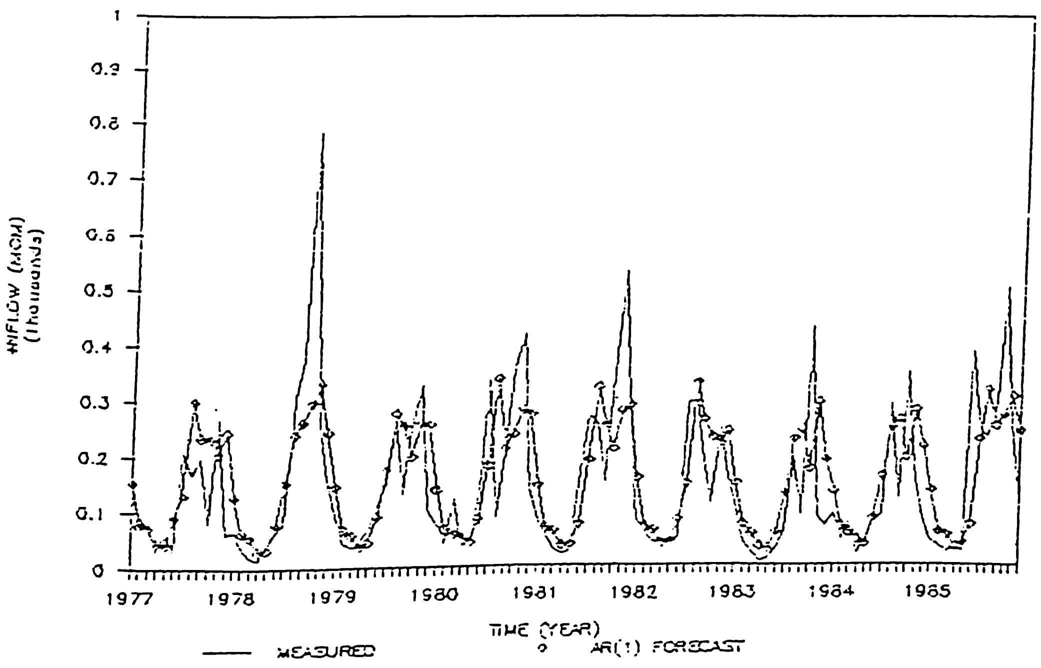
$$Q = \begin{bmatrix} \sigma_{wt1}^2 & 0 \\ 0 & \sigma_{wt2}^2 \end{bmatrix}$$

$$R = \begin{bmatrix} \sigma_{vt1}^2 & 0. \\ 0 & \sigma_{vt2}^2 \end{bmatrix}$$

Figure 9. State-space Formulation of ARMAX Model (ARMA process with exogenous variables)

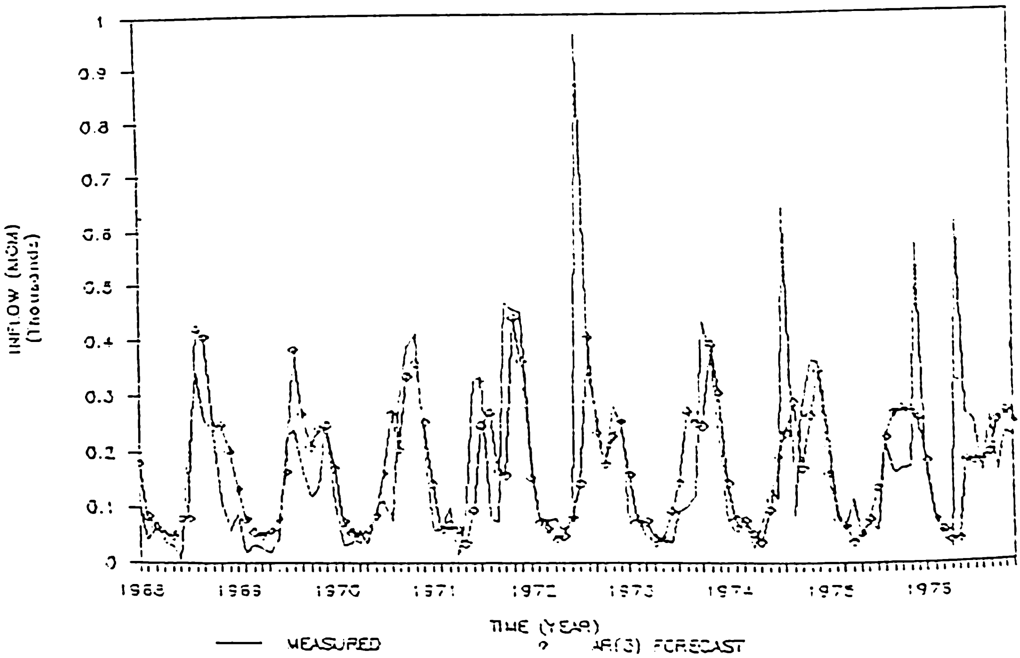


AR(1) Forecast (1968 - 1976)

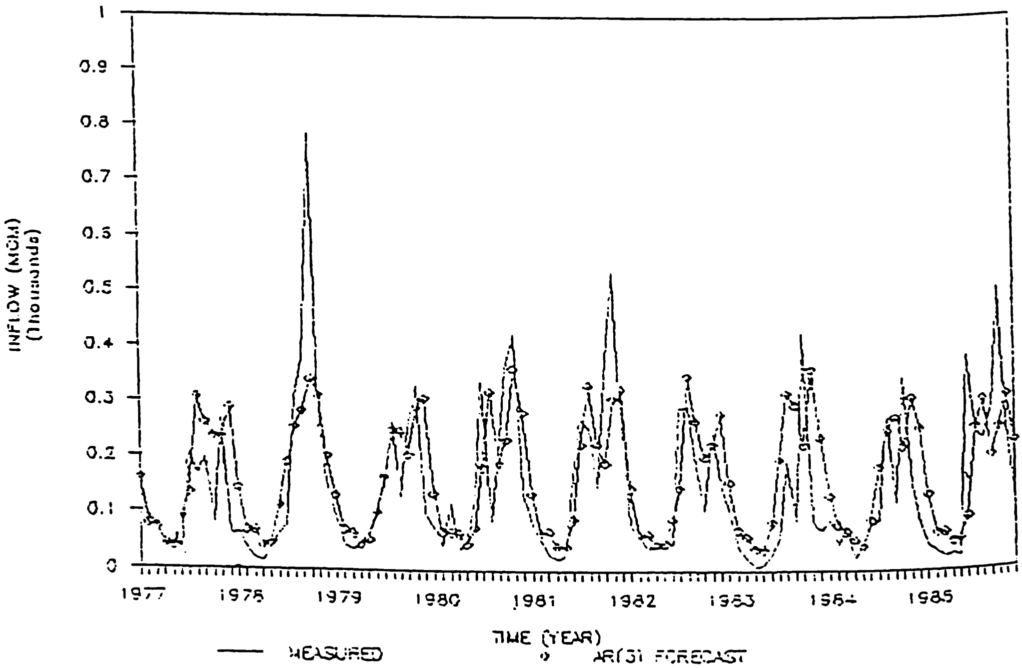


AR(1) Forecast (1977 - 1985)

Figure 10. AR (1) Forecast with State Estimation

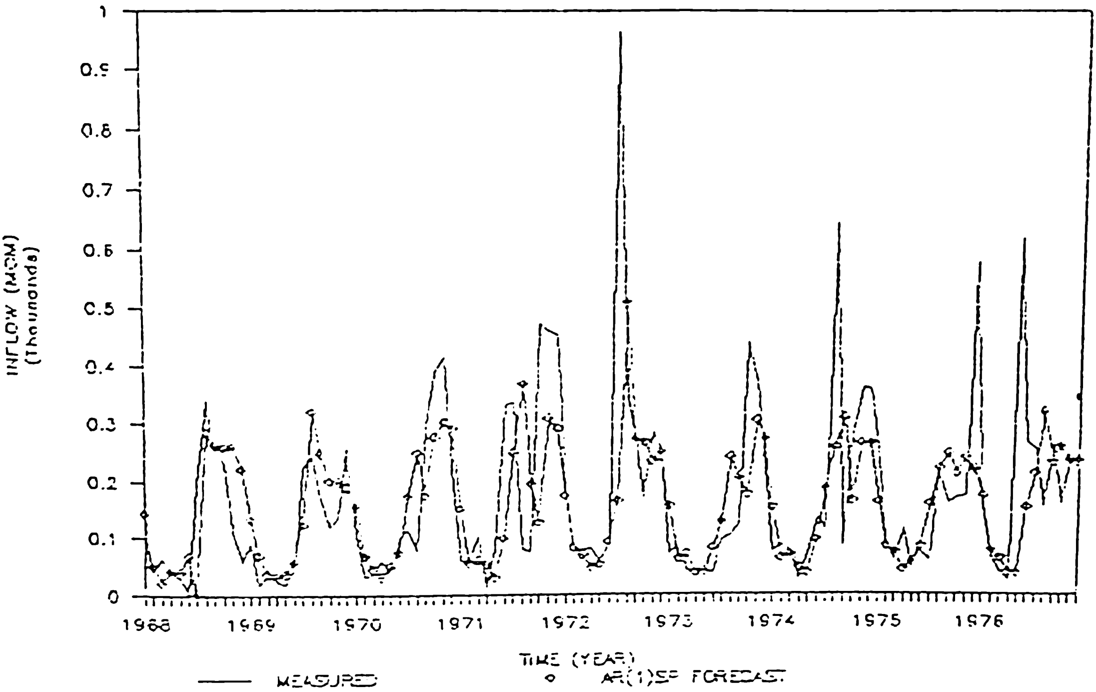


AR(3) Forecast (1968 - 1976)

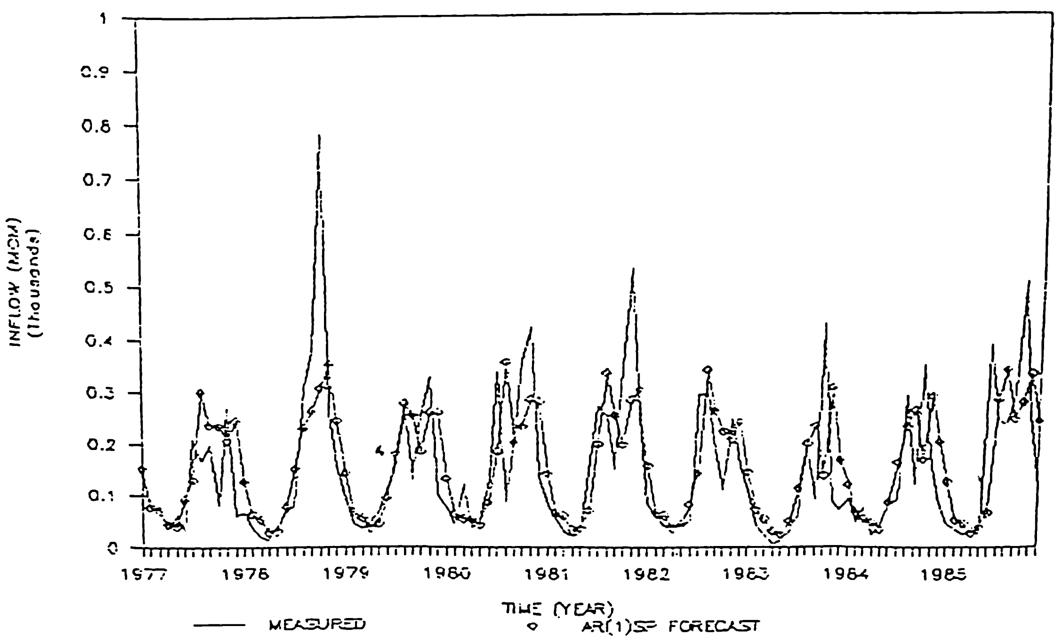


AR(3) Forecast (1977 - 1985)

Figure 11. AR (3) Forecast with State Estimation



AR(1)SP Forecast (1968 - 1976)



AR(1)SP Forecast (1977 - 1985)

Figure 12. AR (1) Forecast with State-Parameter Estimation

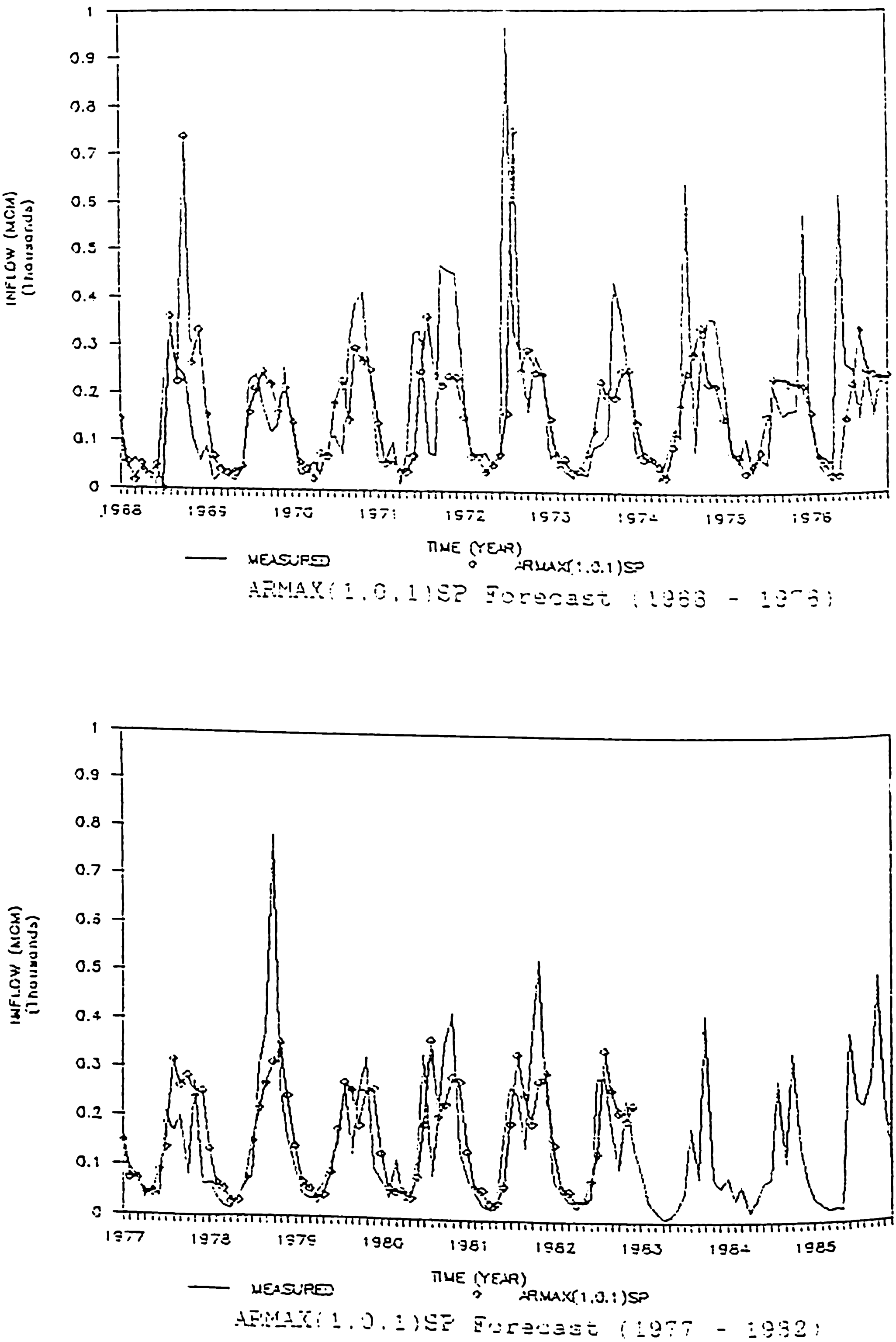
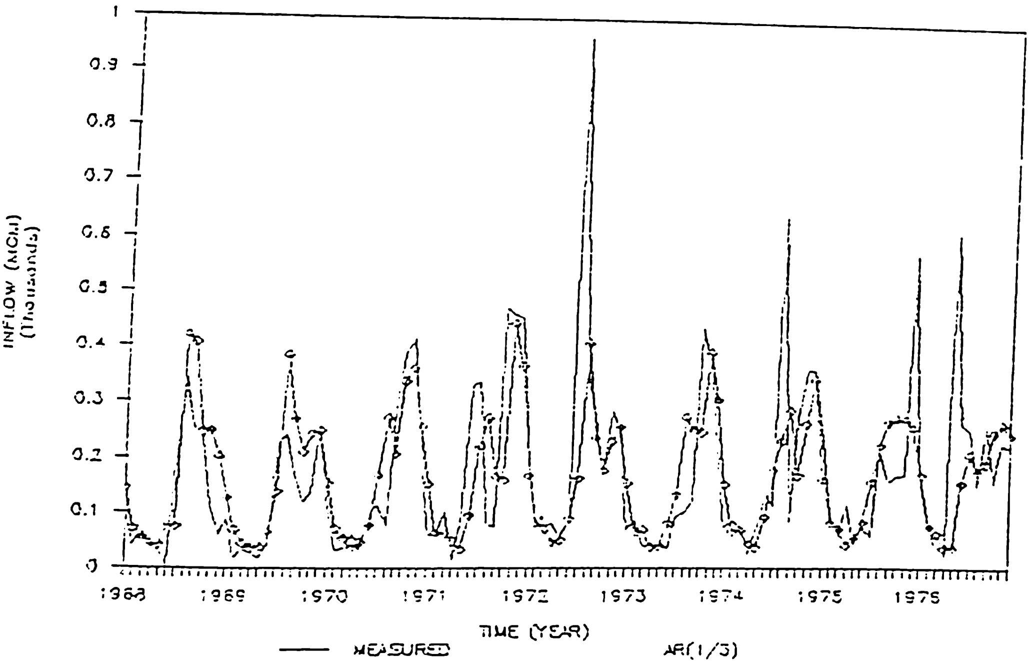
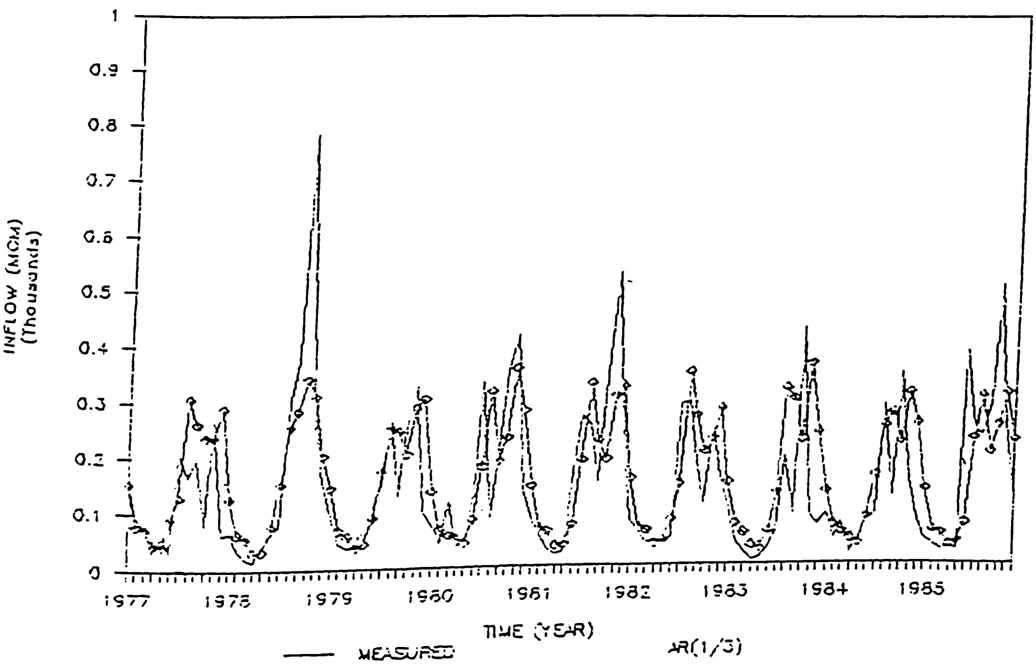


Figure 13. ARMAX (1, 0, 1) Forecast with State-Parameter Estimation



AR(1/3) Forecast (1968 - 1976)



AR(1/3) Forecast (1977 - 1985)

Figure 14. AR (1/3) Forecast

Month	Season	Periodic AR Order $p(v)$	Periodic MA Order $q(v)$	Periodic $\phi_k(v)$	Parameters $\theta_k(v)$
January	1	0	1		$\theta_1(1) = -0.3529$
February	2	1	2	$\phi_1(2) = 0.9135$	$\theta_1(2) = 0.5869$
					$\theta_2(2) = 0.1696$
March	3	0	1		$\theta_1(3) = -0.4584$
April	4	0	2		$\theta_1(4) = -0.4043$
					$\theta_2(4) = -0.2089$
May	5	0	1		$\theta_1(5) = -0.5206$
June	6	2	0	$\phi_1(6) = 0.4481$	
				$\phi_2(6) = 0.0582$	
July	7	1	0	$\phi_1(7) = 0.3935$	
August	8	0	2		$\theta_1(8) = -0.4861$
					$\theta_2(8) = -0.3918$
September	9	0	1		$\theta_1(9) = -0.2937$
October	10	0	1		$\theta_1(10) = -0.4987$
November	11	1	0	$\phi_1(11) = 0.2490$	
December	12	0	3		$\theta_1(12) = -0.6493$
					$\theta_2(12) = -0.4034$
					$\theta_3(12) = 0.0533$

Figure 15. Periodic Order and Parameters of the PARMA Forecasting Model

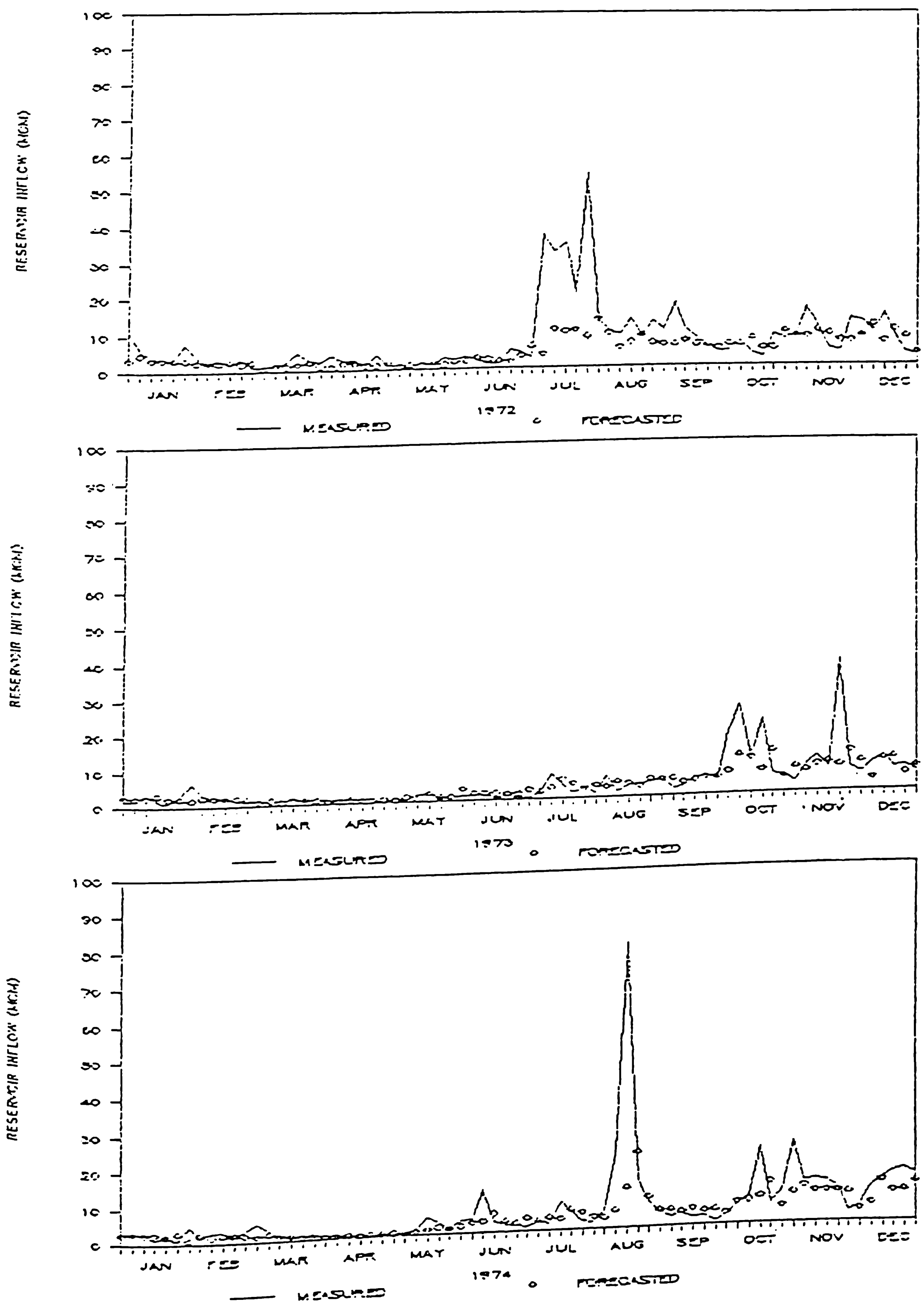
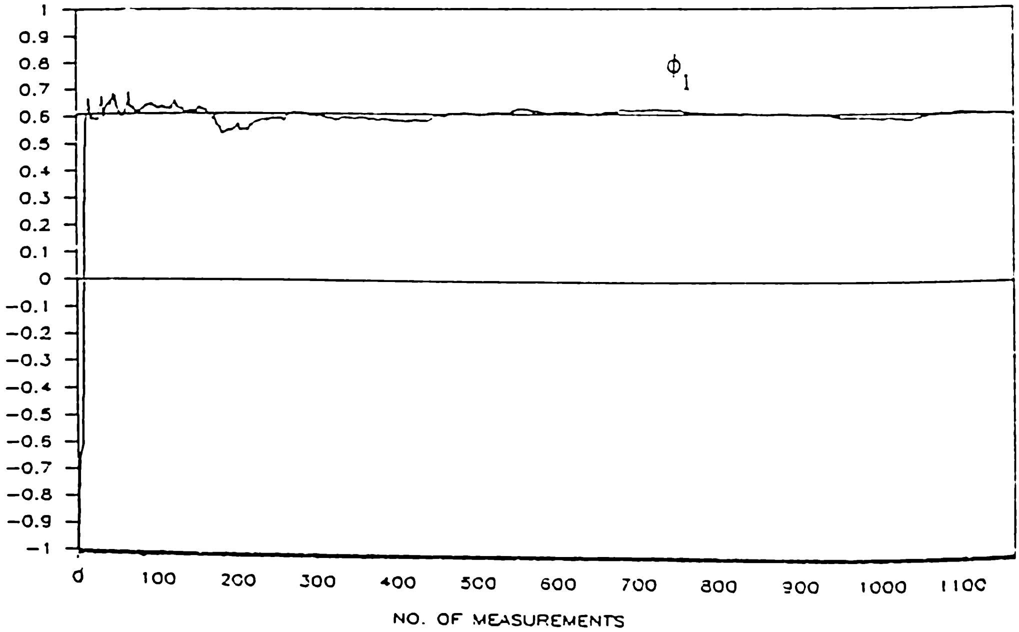
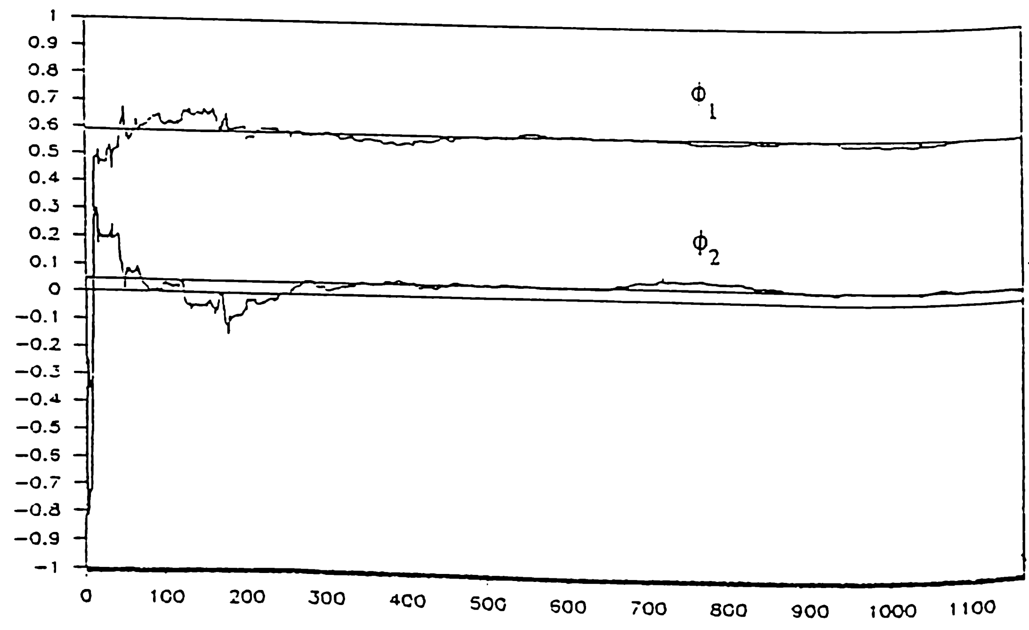


Figure 16. Average 5-day Inflows: Forecast vs. Measured (1972 - 1974)

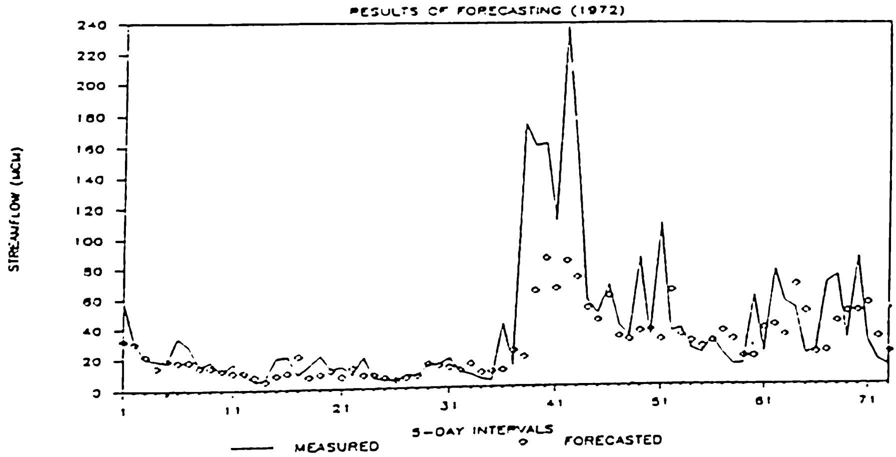


PARAMETER ESTIMATE FROM COMBINED STATE-PARAMETER ESTIMATION [AR(1) MODEL]

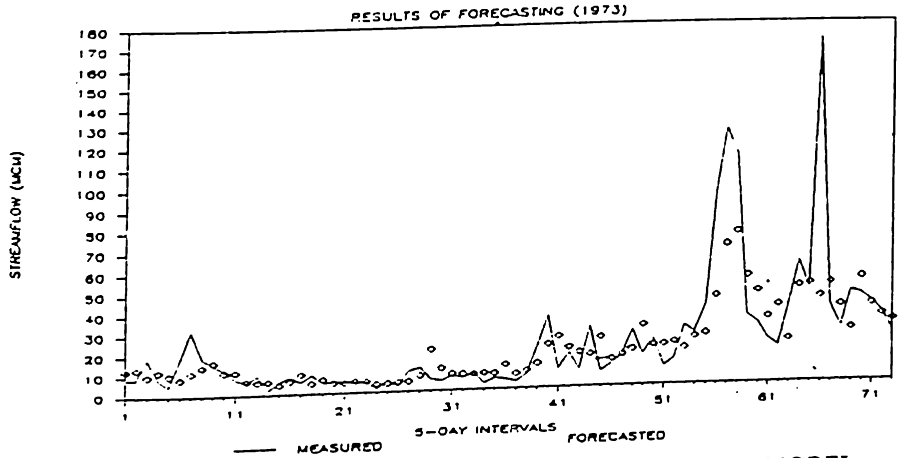


PARAMETER ESTIMATES FROM COMBINED STATE-PARAMETER ESTIMATION [AR(2) MODEL]

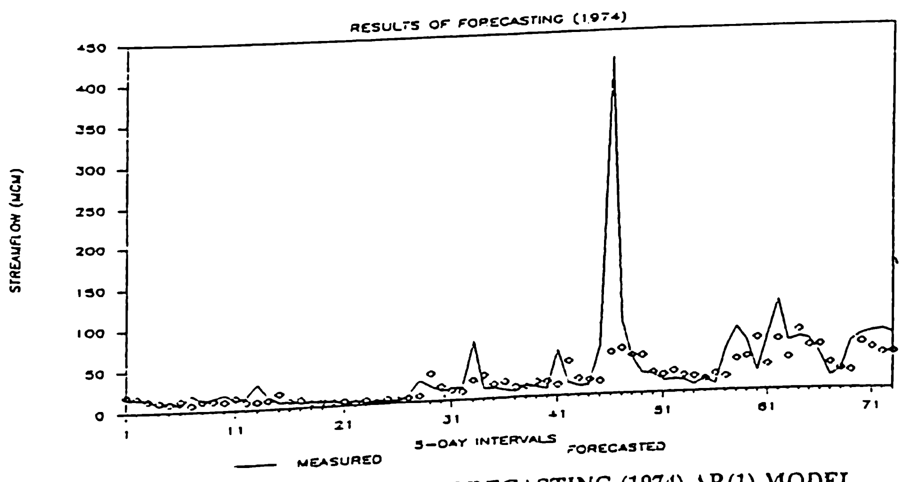
Figure 17. Parameter Estimates from Combined 5-day State-Parameter Estimation



RESULTS OF FORECASTING (1972) AR(1) MODEL

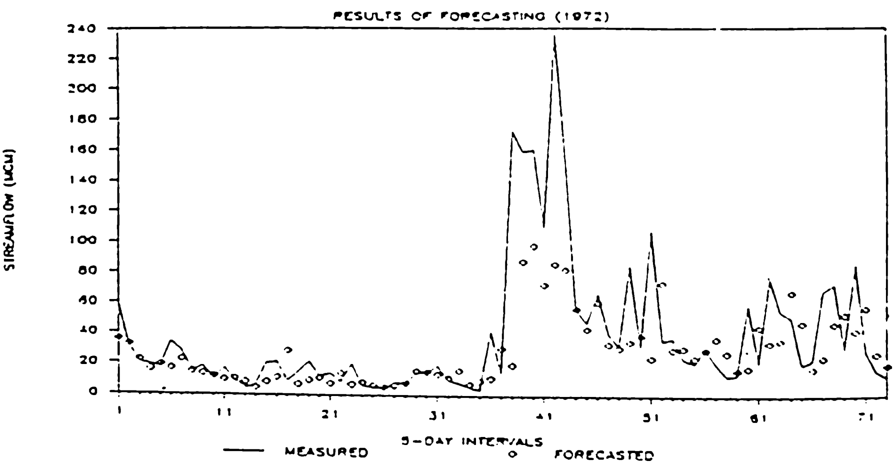


RESULTS OF FORECASTING (1973) AR(1) MODEL

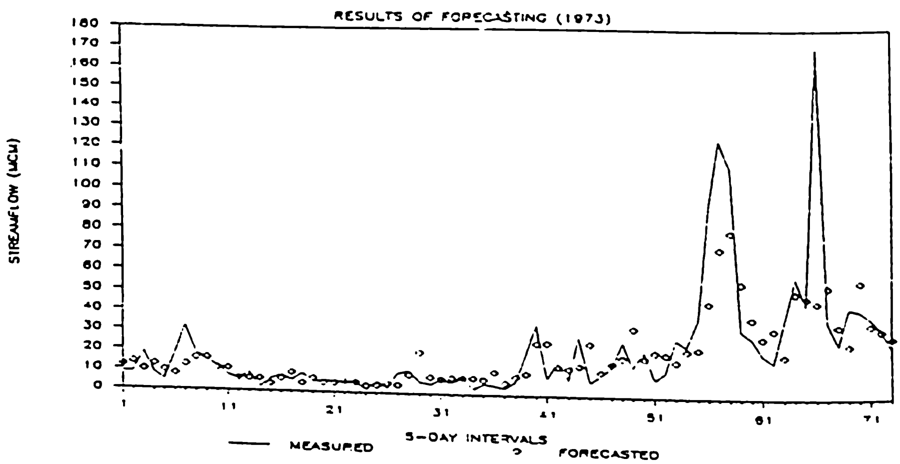


RESULTS OF FORECASTING (1974) AR(1) MODEL

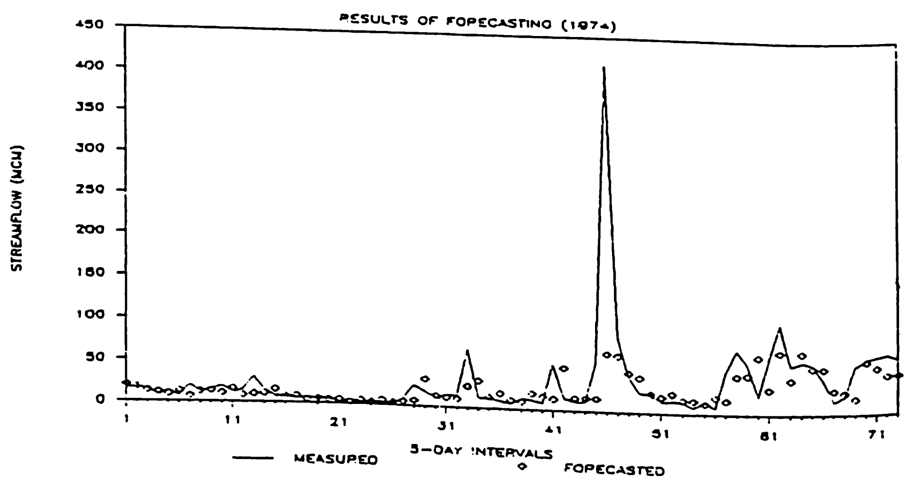
Figure 18. Results of 5-day Flow Forecasting AR (1) Model



RESULTS OF FORECASTING (1972) ARMAX MODEL



RESULTS OF FORECASTING (1973) ARMAX MODEL



RESULTS OF FORECASTING (1974) ARMAX MODEL

Figure 19. Results of 5-day Flow Forecasting, ARMAX Model