"new concepts and definitions are applied to spectrum analysis, FIR digital filter design and the analysis of dynamic frequency response."

Time - Variance Spectrum & Dynamic Frequency Response

by
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ABSTRACT

Based on the natural characteristic of transient signals, the new definitions, time-variance spectrum and dynamic frequency response, are presented. As their applications, the spectrum analyses, the design method of a Finite Impulse Response (FIR) digital filter and frequency responses are used as the examples. Both definitions are beneficial to practical uses in which a short time duration for processing transient signals is required.

INTRODUCTION

The objective of the research is to develop a general method for analysing a time-discrete system which is used in certain cases. In many applications of digital signal processing, there are particular processing objects and special requirements. The digital filter used for the digital relay of a power system and the feed-back unit in a discrete-time control system are the examples of this case. They have the following peculiarities:

1. The processing objects are transient or dynamic signals.
2. Short processing time is required.
3. Sometimes, a fairly low sampling rate has to be used.

One might feel that it is difficult to use common methods for the purposes of design and analysis in the cases mentioned above.

To establish new concepts and develop new definitions, the following work has been done. First, the natural characteristic of lumped and distributed systems is summarized. Second, through the expansions of some classical definitions, the new definitions of time-variance spectrum and dynamic frequency response are derived. And last, the new concepts and definitions are applied to spectrum analysis, FIR digital filter design and the analysis of dynamic frequency response.

DEVELOPED MATHEMATICAL MODEL

The Natural Characteristic of Transient Signals

Before analyzing the natural characteristic of transient signals produced by distributed system, we recall the concept of transients on lumped linear system. For any stable system, the transient

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components must be of the following form:

$$y(t) = \sum_i A_i \exp(-t/\tau_i) + \sum_j A_j\, t \exp(-t/\tau_j) + \sum_k A_k \exp(-t/\tau_k) \mathrm{Sin}(w_k + \phi_k). \qquad (2.1)$$

The three terms at the right-hand side of the equation are the time response with respect to real-single roots of the eigenequation of the system, real-repeated roots and complex-conjugate roots of the eigenequation, respectively.

When an input signal is suddenly applied to the system, the outputs include both the components described in Equation (2.1) and the forms of the original components of the input.

The concept of transients on the lumped linear system was mentioned because in most cases the electromagnetic or non-electromagnetic transient signals have to pass through some lumped circuits before entering a processing unit.

Now, we turn to the analysis of the natural characteristics of electromagnetic transient signal... A typical form of the transient signal produced by a distributed system is:

$$u_t(x,t) = \sum_{n=0}^{\infty} \left(v_n^+ \exp(-a_n x)\cos(\beta_n t - b_n x + p_n) + v_n^- \exp(a_n x)\cos(\beta_n t + b_n x + q_n)\right)\exp(-\alpha_n t). \qquad (2.2)$$

In fact, Equation (2.2) is the expression of transient voltages of a symmetrical fault of a distributed power transmission system [6]. It appears in the form of a travelling-wave.

When x is given, as the distance from the observed point to the reference point, then Equation (2.2) becomes,

$$u_t(t) = \sum_{n=0}^{\infty} v_n \cos(\beta_n t - \Theta_n)\exp(-\alpha_n t). \qquad (2.3)$$

Where,

$$v_n = \sqrt{\begin{array}{l}(v_n^+ \cos\Theta_{1n} \exp(-a_n x) + v_n^- \cos\Theta_{2n} \exp(a_n x))^2 \\ + (v_n^+ \sin\Theta_{1n} \exp(-a_n x) + v_n^- \sin\Theta_{2n} \exp(a_n x))^2\end{array}}$$

$$\Theta_n = \arctan\left[-(v_n^+ \sin\Theta_{1n} \exp(-a_n x) + v_n^- \sin\Theta_{2n} \exp(a_n x))\right] /$$

$$\Theta_{1n} = -b_n x + p_n, \qquad \Theta_{2n} = b_n x + q_n$$

The transient voltage in Equation (2.3) is no longer in the form of a travelling-wave. Instead, it is dependent on the natural frequencies of a system, $(\alpha_n + j\beta_n)$. This means Equation (2.3) describes the natural characteristic of a transient voltage produced by a distributed transmission system.

In a more complex case, for example, an unsymmetrical fault, the fundamental form of transient signal is the same as in Equation (2.3), with a different natural frequency group.

Time-Variance Spectrum

The definition of the time-variance spectrum of the signal described by Equation (2.3) is given as follows:

$$C(\beta_n, t) = v_n \exp(-\alpha_n t - j\Theta_n). \qquad (2.4)$$

$$n = 0, 1, 0\ 2, \ldots\ldots\ldots\ldots\ldots \infty$$

That is,

$$|C(\beta_n, t)| = |v_n| \exp(-\alpha_n t)$$

and

$$\arg[C(\beta_n, t)] = -\Theta_n .$$

These are called time-variance spectra because the coefficient changes with time. For example,

$$|C(\beta_n, 0)| = |v_n|$$

and

$$|C(\beta_n, \infty)| = 0 .$$

In fact, a new definition, the dynamic exponential Fourier coefficient, can be obtained by expanding the definition of exponential Fourier coefficient [2]. That is,

$$B(-\alpha_n + j\beta_n) = \frac{1}{T_n}\int_0^{T_n} u_t(t) \exp(\alpha_n - j\beta_n)t \, dt .$$

$$-\infty < \beta_n < \infty$$

If $u_t(t) = A \exp(-\alpha_n) \cos \quad \cos\beta_n t$, then

$$B(-\alpha_n + j\beta_n) = \frac{1}{T_n}\int_0^{T_n} A \cos\beta_n t \exp(-j\beta_n t) \, dt$$

$$-\infty < \beta_n < \infty$$

The result of the integration is shown in Figure 2.1.

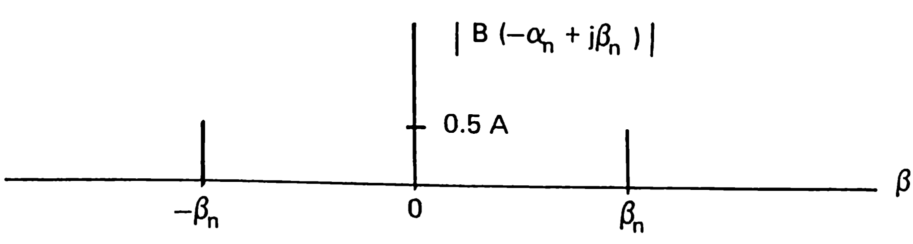


Figure 2.1. The Spectrum of u_t (t)

Comparing the dynamic exponential Fourier coefficient to the coefficient of time-variance spectrum, we can get

$$C(\beta_n, 0) = 2B(-\alpha_n + j\beta_n).$$

The definition of time-variance spectrum will be the theoretical basis of the important fact that there is no aliasing problem for a fairly low sampling rate when the analytical method of the time-variance spectrum is applied.

Dynamic Frequency Response

In this section, the dynamic frequency response will be given by the following derivation. Considering a convolution,

$$Y(n) = \sum_{k=-\infty}^{\infty} h(k) \quad x(n-k) = h(n) * x(n) .$$

Let $x(n) = \exp((-\alpha_d + j\beta_d)n)$.

Where,

$(-\alpha_d + j\beta_d) = (-\alpha_n + j\beta_n)T = s_n T$.

We call $(-\alpha_d + j\beta_d)$ dynamic digital complex frequency. T is sampling interval and s_n is called natural frequency.

So, we have

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \exp((-\alpha_d + j\beta_d)(n-k)) =$$

$$= \exp((-\alpha_d + j\beta_d)n) \sum_{k=-\infty}^{\infty} h(k) \exp((\alpha_d - j\beta_d)k).$$

If we define

$$H(\exp(-\alpha_d + j\beta_d)) = \sum_{k=-\infty}^{\infty} h(k) \exp((\alpha_d - j\beta_d)k).$$

then,

$$y(n) = H(\exp(-\alpha_d + j\beta_d)) \exp((-\alpha_d + j\beta_d)n),$$

If h(k) is the unit-sample response of a discrete system, then $H(\exp(-\alpha_a + j\beta_a))$ is defined as the dynamic digital frequency response of the discrete system. This definition enables us to do dynamic analysis because $H(\exp(-\alpha_d + j\beta_d))$ describes the response to the input signals which are of time-variance spectrum.

APPLICATIONS

Spectrum Analysis

Consider a transient voltage signal as follows:

$$\begin{aligned} u_t(t) = &- 0.0197U_m \exp(-41.2t) + 0.720U_m \exp(-20.5t) \\ &\cos(1451.5t - 0.012) + 0.206U_m \exp(-14.8t) \\ &\cos(1921.3t + 3.137) + 0.0382U_m \exp(-20.5t) \\ &\cos(2957.1t + 3.138) + \dots \end{aligned} \qquad (3.1)$$

Its time-variable spectrum is shown in Figure 3.1.

On the other hand, if we assume the same signal described by Equation (3.1) to be time limited in T_o, say, three times the fundamental frequency, $T_o = 2\pi/(3w_o) = 1/180$ sec., then, repeat it in time with period T_o so that the Fourier series can be obtained, the exponential Fourier coefficient is

$$C(nw_o) = \frac{1}{T_o} \int_0^{T_o} u_y \quad u_t(t) \exp(-jnw_o t)\, dt. \qquad -\infty < n < -\infty \qquad (3.2)$$

The Fourier line spectrum is shown in Figure 3.2.

Comparing Figure 3.1 with Figure 3.2., the conclusion can be made that the series described by Figure 3.1 converges much faster than that described by Figure 3.2. This means that the former is more accurate than the latter when the same first few terms are taken for expressing the same signal. The most important significance is that the Nyquist Sampling Theory is much easier to be satisfied in the former case than in the latter case. In other words, there is still no aliasing problem when a fairly low sampling rate is used for the former case.

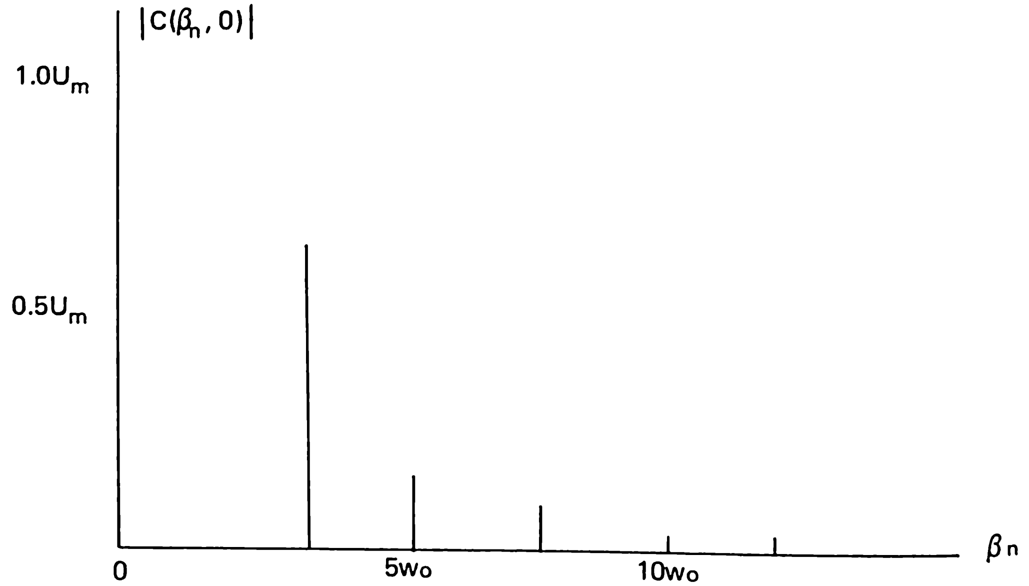


Figure 3.1. Time-Variance Spectrum When t = 0

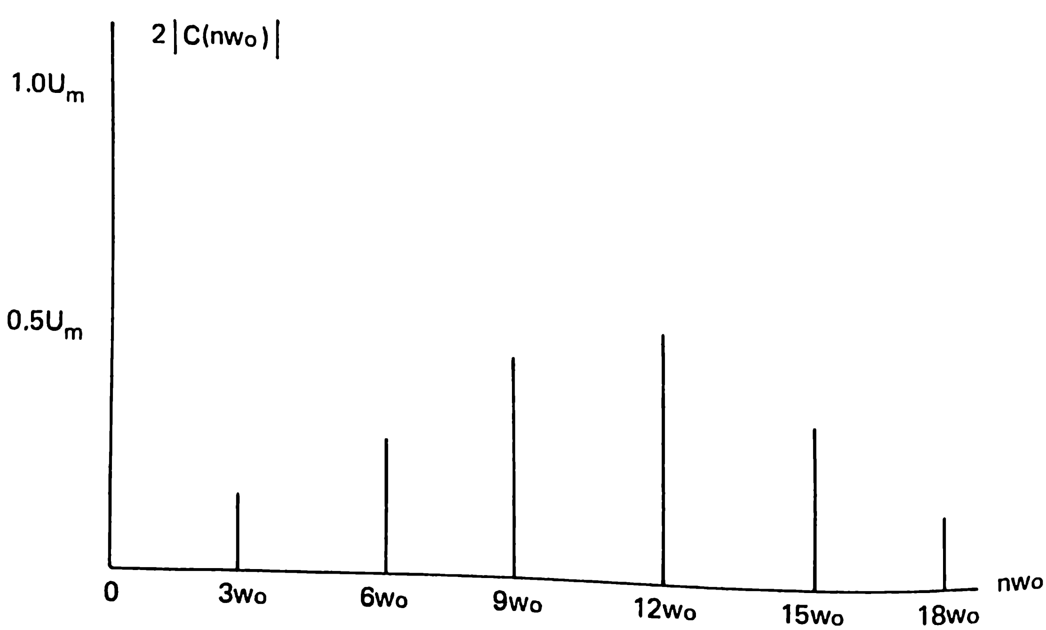


Figure 3.2. Spectrum of Fourier Series

"FIR" Digital Filter Designed by Curve Fitting Method

In many cases, a digital filter has to be used for picking up the component of a fundamental frequency from a transient signal during a very short processing time. The digital filter used for the digital relay of a power system is the example of these cases. At present, the transient input signal of a digital filter (algorithm) is usually considered as a random signal [3] or expressed as a Fourier Series [4]. As shown in the preceding section, the Fourier line spectrum has a wide distribution. In the curve fitting method, if the transient signal is considered as a Fourier Series, the number of order of a FIR (Finite Impulse Response) digital filter must be very high because so many Fourier's components have to be involved. In this section, the natural characteristic of a transient signal which is of time-variance spectrum will be used for designing a digital filter by curve fitting method. Based on the analysis discussed the section on the National Characteristic of Transient Signals, it is reasonable to assume that the transient signal which enters the digital filter is in the following form:

$$v(t) = K_v \sin(w_0 t + \Theta_v) + K_{v0} \exp(-\alpha_0 t) + \sum_{n=1}^{2} K_{vn} \exp(-\alpha_n t) \sin(\beta_n t + \Theta_{vn}) , \qquad (3.3)$$

By using trigonometric substitutions and Taylor Series Expansion with respect to β_n instead of t, we obtain the following expressions.

$$V(t) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n + \ldots + a_{11} x_{11} = \sum_{n=1}^{11} a_n x_n , \qquad (3.4)$$

Where,

$a_1 = \sin w_0 t$, $a_2 = \cos w_0 t$, $a_3 = \exp(-\alpha_0 t)$,

$a_4 = \exp(-\alpha_1 t) \sin w_1 t$, $a_5 = \exp(-\alpha_1 t)\ t \cos w_1 t$,

$a_6 = \exp(-\alpha_1 t) \cos w_1 t$, $a_7 = -\exp(-\alpha_1 t)\ t \sin w_1 t$,

$a_8 = \exp(-\alpha_2 t) \sin w_2 t$, $a_9 = \exp(-\alpha_2 t)\ t \cos w_2 t$,

$a_{10} = \exp(-\alpha_2 t) \cos w_2 t$, $a_{11} = -\exp(-\alpha_2 t)\ t \sin w_2 t$,

$X_1 = K_v \cos\Theta_v$, $X_2 = K_v \sin\Theta_v$, $X_3 = K_{v0}$,

$X_4 = K_{v1} \cos\Theta_{v1}$, $X_5 = (\beta_1 - w_1) K_{v1} \cos\Theta_{v1}$,

$X_6 = K_{v1} \sin\Theta_{v1}$, $X_7 = (\beta_1 - w_1) K_{v1} \sin\Theta_{v1}$,

$X_8 = K_{v2} \cos\Theta_{v2}$, $X_9 = (\beta_2 - w_2) K_{v2} \cos\Theta_{v2}$,

$X_{10} = K_{v2} \sin\Theta_{v2}$, $X_{11} = (\beta_2 - w_2) K_{v2} \sin\Theta_{v2}$.

w_1 , w_2 — expansion points..

At sampling times $t_1, t_2, \ldots, t_m$ ($m \geq 11$), the sampling values can be expressed in matrix form:

$$[V] = [A]\ [X] , \qquad (3.5)$$

It is evident that matrix [A] is a constant matrix because the sampling interval T and w_n ($n = 0, 1, 2$) are given and the natural frequencies, $(\alpha_n + j\beta_n)$, are known by the transient analysis of the system.

By the Least-Squares procedure,

$$[A]^T [A] \; [X] = [A]^T \; [V] \; , \tag{3.6}$$

Thus,

$$[X] = [B] \; [V] \; , \tag{3.7}$$

Where,

$$[B] = \{[A]^T \; [A]\}^{-1} \; [A]^T = \begin{bmatrix} b_{11} & b_{12} & & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ b_{111} & b_{112} & \cdots & b_{11m} \end{bmatrix}$$

solving for the variables in which we are interested in

$$x_1 = K_v \cos\Theta_v = \sum_{n=1}^{m} b_{1n} v_n \; , \tag{3.8}$$

$$x_2 = K_v \text{ so } \sin\Theta_v = \sum_{n=1}^{m} b_{2n} v_n \; , \tag{3.9}$$

Finally, we obtain the magnitude and phase angle of the fundamental component.

$$K_v = \sqrt{(K_v \cos\Theta)^2 + (K_v \sin\Theta)^2} = \sqrt{x_1^2 + x_2^2} \; , \tag{3.10}$$

$$\Theta_v = \arctan\,(\sin\Theta_v \,/\, \cos\Theta_v) = \arctan\,(x_2 \,/\, x_1) \; . \tag{3.11}$$

Frequency Responses

For the convenience of expression, a practical example is discussed below.

The following parameters are given:

w_o = 377 (rad./sec.) , w_1 = 1508 (rad./sec.) ,

w_2 = 3016 (rad./sec.) , α_0 = 63.43 (rad./sec.)

$\alpha_1 = \alpha_2 = \alpha = 32.69$ (rad./sec.) m = 24.

f_s = 1800.0 Hz,

By the design method mentioned in the preceding section, the magnitude of a fundamental frequency component can be obtained from the mth output of the FIR digital filter. During the period $0 \leq k \leq m$, however, the outputs of the digital filter should be that

$$x_1(k) = \sum_{n=1}^{k} b_{1n} v_n \; , \qquad x_2(k) = \sum_{n=1}^{k} b_{2n} v_n \; .$$

$$0 \leq k \leq m$$

Where,

b_{11} = −0.04768, b_{21} = 0.11040 ,

b_{12} = 0.00081 , b_{22} = 0.03648 ,

b_{13}	=	0.01370 ,	b_{23}	=	0.07404 ,
b_{14}	=	0.02268 ,	b_{24}	=	0.06591 ,
b_{15}	=	0.02682 ,	b_{25}	=	0.05914 ,
b_{16}	=	0.01981 ,	b_{26}	=	0.08817 ,
b_{17}	=	0.03389 ,	b_{27}	=	0.07316 ,
b_{18}	=	0.07613 ,	b_{28}	=	–0.01591 ,
b_{19}	=	0.09339 ,	b_{29}	=	–0.07934 ,
b_{110}	=	0.06638 ,	b_{210}	=	–0.07202 ,
b_{111}	=	0.04568 ,	b_{211}	=	–0.06550 ,
b_{112}	=	0.05040 ,	b_{212}	=	–0.08779 ,
b_{113}	=	0.04273 ,	b_{213}	=	–0.09336 ,
b_{114}	=	0.02247 ,	b_{214}	=	–0.08579 ,
b_{115}	=	0.02184 ,	b_{215}	=	–0.10119 ,
b_{116}	=	0.01391 ,	b_{216}	=	–0.10963 ,
b_{117}	=	–0.04285 ,	b_{217}	=	–0.06975 ,
b_{118}	=	–0.10401 ,	b_{218}	=	–0.01779 ,
b_{119}	=	–0.10915 ,	b_{219}	=	–0.00272 ,
b_{120}	=	–0.08886 ,	b_{220}	=	–0.00932 ,
b_{121}	=	–0.09035 ,	b_{221}	=	–0.00795 ,
b_{122}	=	–0.09144,	b_{222}	=	0.00223 ,
b_{123}	=	–0.08305 ,	b_{223}	=	0.02959 ,
b_{124}	=	–0.11362 ,	b_{224}	=	0.08855 .

According to the definition of the convolution of two sequences, the unit-sample response of the FIR digital filter are as follows:

$$h_1(n) \equiv B_{1n} = b_{1(m-n)}, \tag{3.12}$$

$$h_2(n) \equiv B_{2n} = b_{2(m-n)} \tag{3.13}$$

$$0 \leq n \leq m-1$$

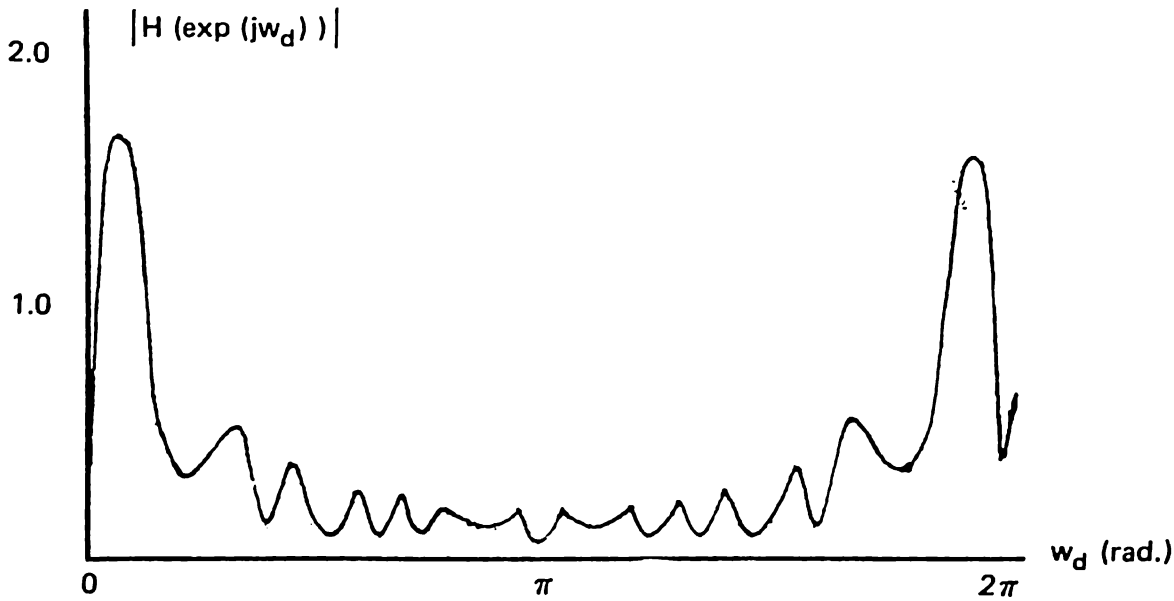
For h_1 (n), the digital frequency response is

$$H(\exp(jw_d)) = \sum_{k=-\infty}^{\infty} h_1(k) \exp(-jw_d k) . \tag{3.14}$$

And the dynamic digital frequency response is

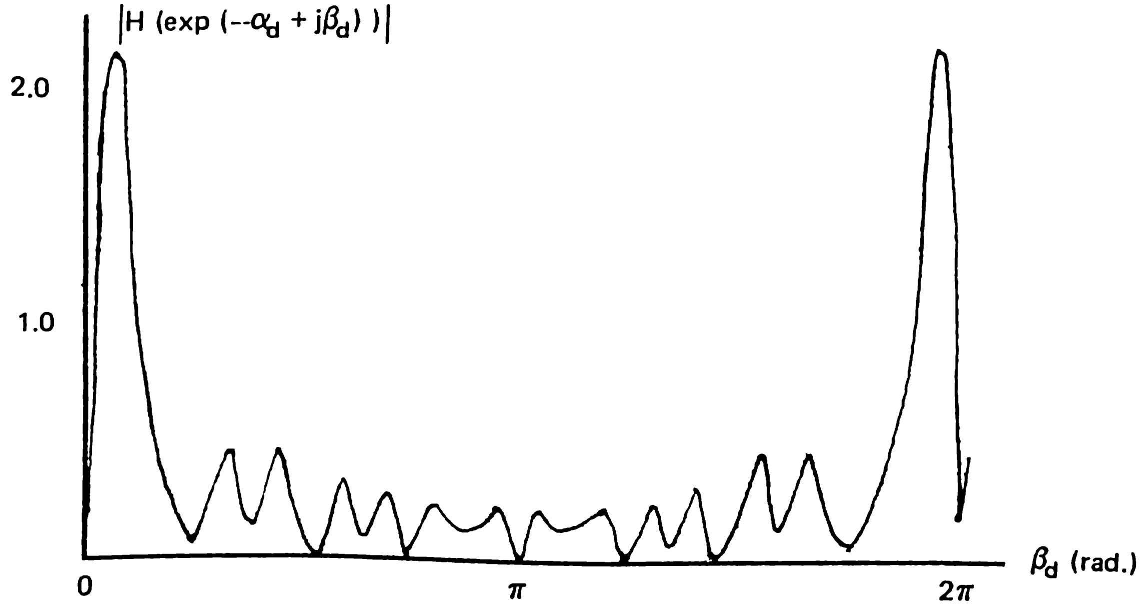
$$H(\exp(-\alpha_d + j\beta_d)) = \sum_{k=-\infty}^{\infty} h_1(k) \exp((\alpha_d - j\beta_d) k). \tag{3.15}$$

Both frequency responses are shown in Figure 3.3 and Figure 3.4.



f_s = 1800 Hz, m = 24.

Figure 3.3. Digital Frequency Responses



α = 32.69 rad./sec., f_s = 1800 Hz, m = 24.

Figure 3.4. Dynamic Digital Frequency Response

It is important to point out that:

1. The digital frequency response H (exp (jw_d)) describes the response to the input signals which are considered as in the form of Fourier Series; while the dynamic digital frequency response H (exp ($-\alpha_d + j\beta_d$)) describes the response to the input signals which are of the time-variance spectra.
2. The larger the value of α is, the bigger the difference between both frequency responses.

3. It is exactly our desire that the dynamic digital frequency responses (in Figure 3.4) are zero at the expansion points.
4. In this example, the length of data window is 13.3 (ms). This means that the outputs of the digital filter are supposed to be the same values (constants) from this time point up ($t \geq 13.3$ ms). We just shift the data window if the repeat calculations are required.
5. Comparing to other design method [2], this method produces a low order (24th order) FIR filter with a good band-pass characteristic. So it is beneficial to those uses in which a short time duration for processing transient signals is required.

CONCLUSIONS AND RECOMMENDATIONS

The important contribution of this research is the applications of both concepts, time-variance spectrum and dynamic frequency response, to the design and analysis of a FIR digital filter and to the spectrum analysis of transient signals. By using the analysis method of time-variance spectrum, we have seen that the property of concentrated spectra makes the Nyquist Sampling Theory easily satisfied so that there is no aliasing problem even under a fairly low sampling rate. When the natural characteristic of transient signals which are described by time-variance spectrum is applied to the design of a FIR digital filter, what we get is a low order digital filter with a very narrow band-pass characteristic.

As far as the dynamic digital frequency response is concerned, it provides a direct method for measuring the properties of a discrete-time system in dynamic frequency-domain. One suggestion here is that the dynamic frequency response could be used as a criterion for improving or optimizing the parameters of a time-discrete system by a proper method, such as computer iterative method.

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REFERENCES

[1] L. R. RABINER and B. GOLD. *Theory and Application of Digital Signal Processing.* Prentice Hall, Inc., N. J., 1975.

[2] A. V. OPPENHEIM and R. W. SCHAFER. *Digital Signal Processing.* Prentice Hall, Inc., N. J., 1975.

[3] N. G. JOHNSON, M. S. SACHDEV and H. C. WOOD. Use of Kalman Filtering Technique for Power System Protection – An Insight. Power Systems Research Group, University of Saskatchewan, Canada, 1984.

[4] M. S. SACHDEV. A New Algorithm for Digital Impedance Relay. Power Systems Research Group, University of Saskatchewan, Canada, 1983.

[5] L. R. RABINER. "Linear Program Design of Finite Impulse Response Digital Filter" *IEEE,* Trans. Audio Electroacoust, Vol. 20, No. 4, October 1972.
Filter."

[6] JINGKE CHEN. "Analytic Solution of Electromagnetic Transient In a Polyphase Transmission System." *Philippine Engineering Journal,* Vol VIII, No. 2, December 1987.