"The method uses the concept of direct linkage constraints and motion parameters based on displacements."

A New Method for the Kinematic Analysis of Planar Four-Bar Mechanisms

by

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ABSTRACT

A convenient method for the analysis of planar four-bar mechanisms is achieved by applying the concept of direct linkage constraints and motion parameters based on the displacements of the mechanism. Vectors are used extensively to make the derivation simpler and to make resulting analysis equations more compact. The new method also eliminates the need to determine intermediate motion parameters. Thus, only the motion parameters of interest are obtained directly and explicitly. Higher order motion parameters are also derived from the basic linkage constraints. The formulation of the method is shown and applied to the analysis of a crank-rocker and a slider-crank mechanism.

INTRODUCTION

The planar four-bar has been extensively studied in the past and also continues to be scrutinized up to the present. The apparent simplicity of the planar four-bar conceals its potential applications and also the complexity of analyzing the mechanism. Planar mechanisms have been analyzed using graphical methods requiring the use of drawing instruments and also by the so-called analytical methods like algebraic methods, complex number methods and also vector methods. Analysis is carried out through the development of kinematic relations by analytical means and computations based on these relations. The reader is referred to references (1), (2) and (3) for details of the graphical and analytical methods.

In this paper, a novel method for analyzing four-bars will be presented. The method uses vectors exclusively together with the common scalar (dot) and vector (cross) product operations. Only an introductory background in vectors is required for its understanding.

This new approach is the result of a generalized pair constraint approach that was developed for spatial mechanisms as presented in reference (4). It is interesting to note that the method was originally applied for three-dimensional mechanisms before application to planar mechanisms.

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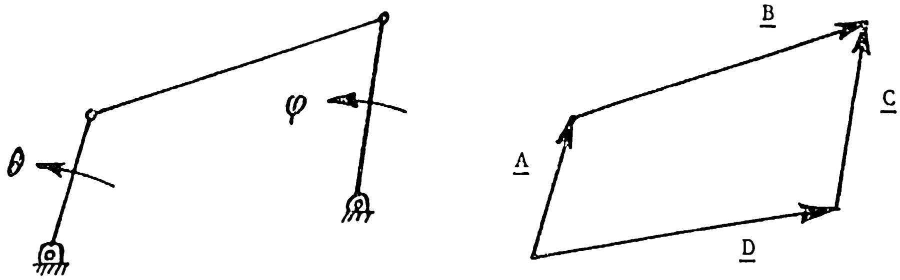

Figure 1. The Crank-Rocker with its Vector and Scalar Representations

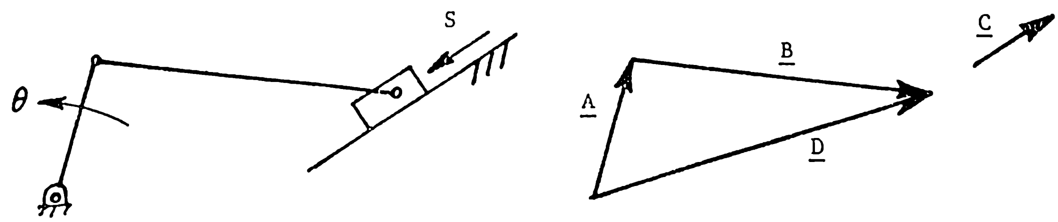

Figure 2. The Slider-Crank with its Vector and Scalar Representations

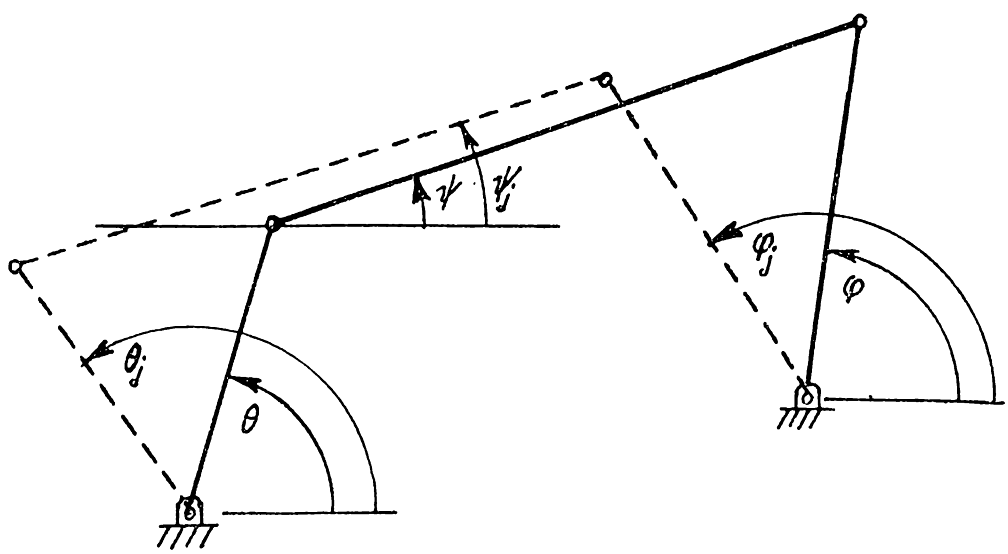

Figure 3. The Crank-Rocker with its Motion Parameters Based on a Reference Axis

THE PLANAR FOUR-BAR MECHANISM

Figure 1 shows a crank-rocker while Figure 2 shows the slider-crank. In the same figures are also shown the vectors and scalars that represent the mechanism and its motion parameters. In the usual or classical methods of analysis of these four-bars, the mechanisms are located on a coordinate system and their positions are "absolutely" referred to this coordinate system. Figure 3 shows these angles. θ, θ_j, ϕ, ϕ_j, ψ, and ψ_j as measured from the horizontal axis. In the case of the crank-rocker shown in Figures 1 and 3, the motion parameters of interest are θ and ϕ. The angle ψ is an example now of an intermediate motion parameter.

THE VECTOR-LINKAGE CONSTRAINT METHOD

The new method of kinematic analysis – displacement, velocity, and acceleration – will now be shown for the four-bar crank-rocker. The reader can compare the new method with those in references (1), (2) and (3) and appreciate the new concept and the differences. Figure 1 shows the vectors $\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{D}$ describing a unique four-bar in its *initial* position. This initial position is simply the position of the four-bar at which the vectors $\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{D}$ are defined. As the first departure from the usual methods, the motion parameters considered are *displacement angles referred from the initial position* and not from one of the axes of the reference coordinate system. These angles, θ_j and ϕ_j, are shown in Figure 4. The same figure also shows the displaced vectors $\underline{A}_j$, $\underline{B}_j$, $\underline{C}_j$, and $\underline{D}$ describing the "displaced four-bar." It should be noted at this point that for analysis, all the vectors and input motion parameters (i.e., $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$, θ_j, $\dot{\theta}_j$, $\ddot{\theta}_j$, etc.) are given and the objective is to find the output motion parameter–in this case ϕ_j, $\dot{\phi}_j$, $\ddot{\phi}_j$, etc. The vector loop closure equation of the displaced four-bar can now be written as:

$$\underline{A}_j + \underline{B}_j = \underline{D} + \underline{C}_j \tag{1}$$

$$\underline{B}_j = \underline{D} + \underline{C}_j - \underline{A}_j \tag{2}$$

The expressions for the displaced vectors are now derived from the rotation of vector $\underline{A}$ about an axis perpendicular to the plane by θ_j shown in Figure 5. The vector $\underline{A}_j$ is chosen to be expressed as a function of the rotational displacement θ_j. To do this, it will be written as the sum of its components in the $\underline{A}$ direction and in a direction perpendicular to $\underline{A}$ which is in the $(\underline{k} \times \underline{A})$ direction where $\underline{k}$ is the unit vector perpendicular to the plane. Thus,

$$\underline{A}_j = \underline{A}\cos\theta_j + (\underline{k} \times \underline{A})\sin\theta_j \tag{3}$$

Similarly, the vector $\underline{C}_j$ is the vector $\underline{C}$ rotated by ϕ_j.

$$\underline{C}_j = \underline{C}\cos\phi_j + (\underline{k} \times \underline{C})\sin\phi_j \tag{4}$$

Substituting equations (3) and (4) in (2) will give us

$$\underline{B}_j = \underline{D} + \underline{C}\cos\phi_j + (\underline{k} \times \underline{C})\sin\phi_j - \underline{A}\cos\theta_j - (\underline{k} \times \underline{A})\sin\theta_j \tag{5}$$

Note here that the sign convention of the angles follow the right hand rule. Since our interest is only in the output motion parameter ϕ_j, we need not derive $\underline{B}_j$ as a function of its motion parameter which is ψ_j. We are now ready to look into the *linkage constraint*. This concept of linkage constraint is the recognition of the geometric constraints imposed by the connecting joints or pairs of a link. In this particular case, the

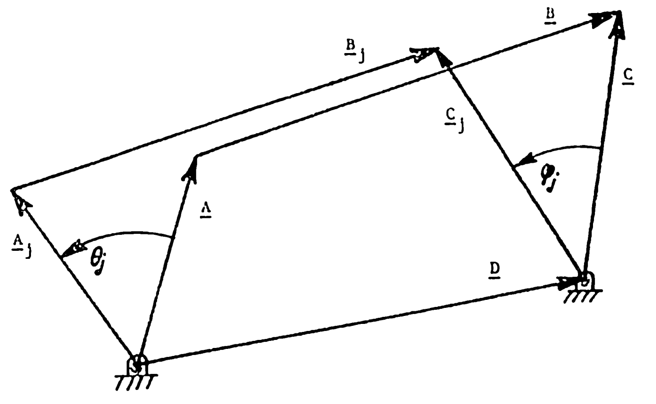

Figure 4. The Crank-Rocker with its jth (displaced) Position Referred from the Initial Position

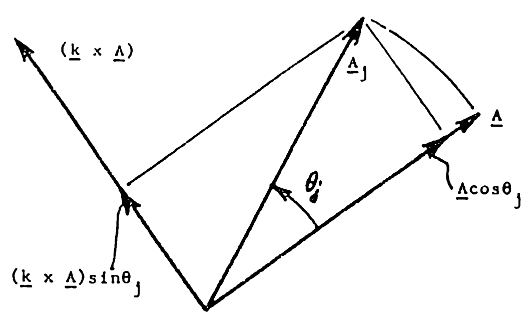

Figure 5. Rotation of Vector $\underline{A}$ by θ_j

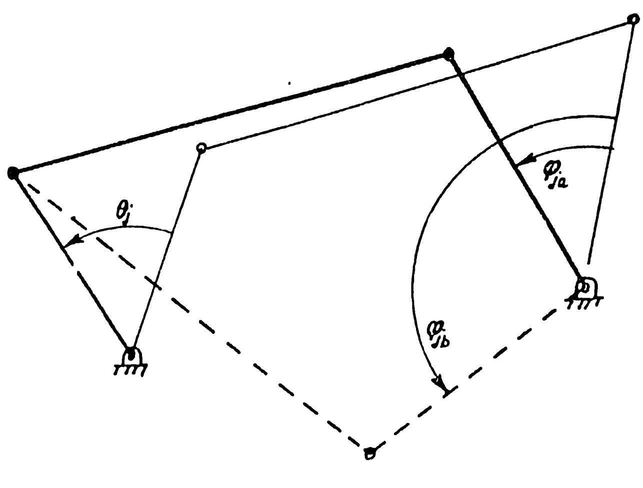

Figure 6. The Two Possible Solutions to φj

constraint of the two revolute ends of the coupler link is that its length is fixed. This coupler link is represented by $\underline{B}$ or $\underline{B}_j$ at its jth position and it is a vector of constant length. This constraint can be represented in an equation – the linkage constraint equation. For the coupler, this is

$$\underline{B} \cdot \underline{B} = |B|^2 \tag{6}$$

and at any other position $\underline{B}_j$,

$$\underline{B}_j \cdot \underline{B}_j = |B|^2 = \underline{B} \cdot \underline{B} \tag{7}$$

Thus, for the crank-rocker (or its other forms like the double crank and the double rocker), we get the linkage constraint equation as:

$$\begin{aligned}\underline{B} \cdot \underline{B} = \; & [\underline{D} + \underline{C}\cos\phi_j + (\underline{k} \times \underline{C})\sin\phi_j - \underline{A}\cos\theta_j - (\underline{k} \times \underline{A})\sin\theta_j] \cdot \\ & [\underline{D} + \underline{C}\cos\phi_j + (\underline{k} \times \underline{C})\sin\phi_j - \underline{A}\cos\theta_j - (\underline{k} \times \underline{A})\sin\theta_j]\end{aligned} \tag{8}$$

Carrying out the dot product operation and taking note of identities and vanishing terms, equation (8) is simplified to

$$\begin{aligned}&\cos\phi_j[(\underline{C} \cdot \underline{D}) - (\underline{A} \cdot \underline{C})\cos\theta_j - \underline{C} \cdot (\underline{k} \times \underline{A})\sin\theta_j] + \\ &\sin\phi_j[(\underline{k} \times \underline{C}) \cdot D - (\underline{k} \times \underline{C}) \cdot \underline{A}\cos\theta_j - (\underline{C} \cdot \underline{A})\sin\theta_j] + \\ &[(\underline{C} \cdot \underline{C} + \underline{D} \cdot \underline{D} + \underline{A} \cdot \underline{A} - \underline{B} \cdot \underline{B})/2 - (\underline{D} \cdot \underline{A})\cos\theta_j - \underline{D} \cdot (\underline{k} \times \underline{A})\sin\theta_j] = 0\end{aligned} \tag{9}$$

Using the trigonometric identities

$$\cos\beta = \frac{1 - \tan^2(\beta/2)}{1 + \tan^2(\beta/2)} \tag{10}$$

$$\sin\beta = \frac{2\tan(\beta/2)}{1 + \tan^2(\beta/2)} \tag{11}$$

We will finally get the quadratic solution to ϕ_j from equation (9) as:

$$\phi_j = 2\tan^{-1}\left[\frac{-b \pm \sqrt{a^2 + b^2 - c^2}}{c - a}\right] \tag{12}$$

$$\begin{aligned}\text{where } a &= (\underline{C} \cdot \underline{D}) - (\underline{A} \cdot \underline{C})\cos\theta_j - \underline{C} \cdot (\underline{k} \times \underline{A})\sin\theta_j \\ b &= (\underline{k} \times \underline{C}) \cdot \underline{D} - (\underline{k} \times \underline{C}) \cdot \underline{A}\cos\theta_j - (\underline{C} \cdot \underline{A})\sin\theta_j \\ c &= (\underline{C} \cdot \underline{C} + \underline{D} \cdot \underline{D} + \underline{A} \cdot \underline{A} - \underline{B} \cdot \underline{B})/2 - (\underline{D} \cdot \underline{A})\cos\theta_j \\ &\quad - \underline{D} \cdot (\underline{k} \times \underline{A})\sin\theta_j\end{aligned} \tag{13}$$

The solution to the displacement problem is quadratic implying two possible solutions. For a given four-bar, the solution is in fact double valued as shown also by the other analytical methods. These two solutions are called "branches" and one can recognize from Figure 6 that although there are really two *possible* solutions, only one set (or branch) can be accepted as the true solution. Whenever imaginary solutions are obtained, it simply means that the mechanism cannot satisfy the input angle θ_j. That is, the input crank cannot move to the position $\underline{A}_j$.

The higher order motion relations are also conveniently obtained by taking the time derivative of the constraint equation, that is,

$$d^n/dt^n(\underline{B}_j \cdot \underline{B}_j) = 0 \qquad (14)$$

The time derivative of $(\underline{B}_j \cdot \underline{B}_j)$ for n = 1 or greater is always zero because the crank length does not change with time.

For the velocity input-output relation,

$$d/dt(\underline{B}_j \cdot \underline{B}_j) = (\dot{\underline{B}}_j \cdot \underline{B}_j) = 0 \qquad (15)$$

$\dot{\underline{B}}_j$ is the first time derivative of $\underline{B}_j$ where the variables of differentiation are θ_j and ϕ_j so that

$$\dot{\underline{B}}_j = d/dt[\underline{D} + \underline{C}\cos\phi_j + (\underline{k} \times \underline{C})\sin\phi_j - \underline{A}\cos\theta_j - (\underline{k} \times \underline{A})\sin\theta_j] \qquad (16)$$

$$\dot{\underline{B}}_j = [-\underline{C}\sin\phi_j\dot{\phi}_j + (\underline{k} \times \underline{C})\cos\phi_j\dot{\phi}_j] - [-\underline{A}\sin\theta_j\dot{\theta}_j + (\underline{k} \times \underline{A})\cos\theta_j\dot{\theta}_j] \qquad (17)$$

or

$$\dot{\underline{B}}_j = \underline{C}_j'\dot{\phi}_j - \underline{A}_j'\dot{\theta}_j \qquad (18)$$

where:

$$\begin{aligned} \underline{C}_j' &= -\underline{C}\sin\phi_j + (\underline{k} \times \underline{C})\cos\phi_j \\ \underline{A}_j' &= -\underline{A}\sin\theta_j + (\underline{k} \times \underline{A})\cos\theta_j \end{aligned} \qquad (19)$$

Note that $\underline{C}_j'$ and $\underline{A}_j'$ are the vectors $\underline{C}_j$ and $\underline{A}_j$ rotated 90 degrees counterclockwise, respectively.

Substituting equation (17) into equation (15) will give us

$$0 = \dot{\underline{B}}_j \cdot \underline{B}_j = (\underline{C}_j'\dot{\phi}_j - \underline{A}_j'\dot{\theta}_j) \cdot \underline{B}_j \qquad (20)$$

or

$$\dot{\phi}_j = \left[\frac{\underline{A}_j' \cdot \underline{B}_j}{\underline{C}_j' \cdot \underline{B}_j}\right]\dot{\theta}_j \qquad (21)$$

For the acceleration relation, we take the second time derivative of $(\underline{B}_j \cdot \underline{B}_j)$ and set it to zero. Thus,

$$d^2/dt^2(\underline{B}_j \cdot \underline{B}_j) = \ddot{\underline{B}}_j \cdot \underline{B}_j + \dot{\underline{B}}_j \cdot \dot{\underline{B}}_j = 0 \qquad (22)$$

The second time derivative of $\underline{B}_j$ is derived as

$$\ddot{\underline{B}}_j = -\underline{C}_j\dot{\phi}_j^2 + \underline{C}_j'\ddot{\phi}_j + \underline{A}_j\dot{\theta}_j^2 - \underline{A}_j'\ddot{\theta}_j \qquad (23)$$

Substituting equation (23) into equation (22) will give us

$$\ddot{\phi}_j = \frac{-\dot{\underline{B}}_j \cdot \dot{\underline{B}}_j - (\underline{A}_j \cdot \underline{B}_j)\dot{\theta}_j^2 + (\underline{A}_j' \cdot \underline{B}_j)\ddot{\theta}_j + (\underline{C}_j \cdot \underline{B}_j)\dot{\phi}_j^2}{(\underline{C}_j' \cdot \underline{B}_j)} \qquad (24)$$

Although the expressions for the input-output relations look lengthy, they can be conveniently programmed since they are explicit.

It should be noted that before any higher order motion parameter can be determined, all the other lower order parameters must be known. Thus, if an acceleration parameter is to be solved for, the displacement and velocity must already be determined.

APPLICATION TO THE SLIDER-CRANK

Figure 7 shows the initial and displaced position of a slider-crank together with the vectors and scalars that describe the mechanism and the motion.

These are:

- $\underline{A}$ – the vector representation of the crank
- $\underline{B}$ – the vector representation of the coupler
- $\underline{C}$ – a *unit vector* defining the direction and sense of the slider
- $\underline{D}$ – a vector locating the initial position of the slider center
- S_j – the linear displacement of the slider. Positive if in the same sense as $\underline{C}$ and negative otherwise

For this slider-crank, the motion parameter of interest is S_j and its derivatives as a function of the input motion θ_j and its derivatives.

As in the crank-rocker, we write the loop closure equation of the displaced mechanism as

$$\underline{A}_j + \underline{B}_j = \underline{D} + S_j\underline{C} \qquad (25)$$

or

$$\underline{B}_j = \underline{D} + S_j\underline{C} - \underline{A}_j \qquad (26)$$

The expression for $\underline{A}_j$ is the same as equation (3) and substituting it again into the linkage constraint of the coupler which is $\underline{B} \cdot \underline{B} = |B|^2 = \underline{B} \cdot \underline{B}$, we can get the expression

$$B \cdot B = [\underline{D} + S_j\underline{C} - \underline{A}\cos\theta_j - (\underline{k} \times \underline{A})\sin\theta_j]$$
$$[\underline{D} + S_j\underline{C} - \underline{A}\cos\theta_j - (\underline{k} \times \underline{A})\sin\theta_j] \qquad (27)$$

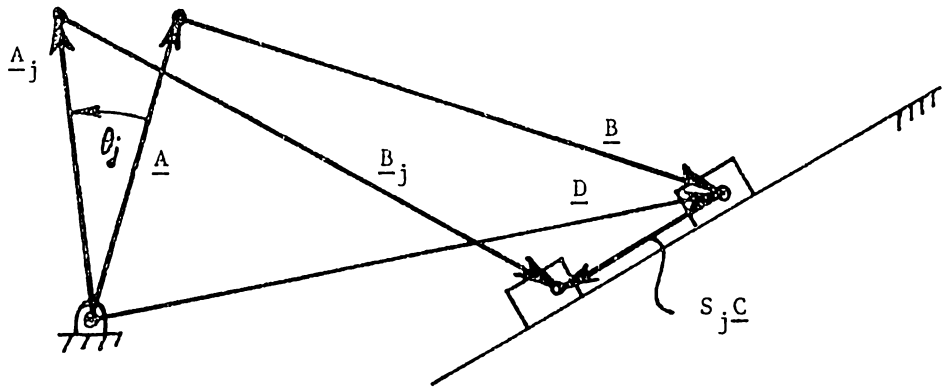

Figure 7. The Initial and Displaced Positions of the Slider-Crank

The linkage-constraint is similar to the crank-rocker but it does not mean that this is the only linkage-constraint that can be used. Other mechanisms will, in general, have different forms of linkage-constraints for any particular link.

Performing the operation in equation (27) will give

$$S_j^2 + 2\underline{C}\,(\underline{D} - \underline{A}_j)S_j + [(\underline{A}\,\underline{A}) - (\underline{B}\,\underline{B}) + (\underline{D}\,\underline{D}) - 2\underline{A}_j\,\underline{D}] = 0 \qquad (28)$$

which is quadratic in S_j, therefore

$$S_j = (-b + \sqrt{b^2 - 4c})/2 \qquad (29)$$

where: $b = 2\underline{C} \cdot (\underline{D} - \underline{A}_j)$

$$c = (\underline{A} \cdot \underline{A}) - (\underline{B} \cdot \underline{B}) + (\underline{D} \cdot \underline{D}) - 2\underline{A}_j \cdot \underline{D} \qquad (30)$$

The velocity and acceleration of the block were derived as follows:

$$\dot{S}_j = (\underline{A}_j' \cdot \underline{B}_j)/(\underline{C} \cdot \underline{B}_j) \qquad (31)$$

$$\ddot{S}_j = \frac{-\dot{\underline{B}}_j \cdot \dot{\underline{B}}_j - (\underline{A}_j \cdot \underline{B}_j)\dot{\theta}_j^2 + (\underline{A}_j'\,\underline{B}_j)\ddot{\theta}_j}{(\underline{C} \cdot \underline{B}_j)} \qquad (32)$$

NUMERICAL EXAMPLE

Consider the four-bar mechanism denoted by the following vectors:

$\underline{A} = 2\,\underline{i} + 3\,\underline{j}$

$\underline{B} = 6\,\underline{i} + 2\,\underline{j}$

$\underline{C} = 2\,\underline{i} + 5\,\underline{j}$

$\underline{D} = 6\,\underline{i}$

We want to obtain the input-output relations for increments of $\theta_j = 20^\circ$ with $\dot{\theta}_j = 1$. Part of the problem also is to determine the range of motion of the mechanism.

The following dot and cross products are first evaluated as:

$$
\begin{aligned}
(\underline{A} \cdot \underline{A}) &= (2)(2) + (3)(3) = 13 \\
(\underline{B} \cdot \underline{B}) &= (6)(6) + (2)(2) = 40 \\
(\underline{C} \cdot \underline{C}) &= (2)(2) + (5)(5) = 29 \\
(\underline{D} \cdot \underline{D}) &= (6)(6) + (0)(0) = 36 \\
(\underline{A} \cdot \underline{C}) &= (2)(2) + (3)(5) = 19 \\
(\underline{A} \cdot \underline{D}) &= (2)(6) + (3)(0) = 12 \\
(\underline{k} \times \underline{A}) &= -a_y \underline{i} + a_x \underline{j} = -3\,\underline{i} + 2\,\underline{j} \\
(\underline{k} \times \underline{C}) &= -c_y \underline{i} + c_y \underline{j} = -5\,\underline{i} + 2\,\underline{j} \\
\underline{C} \cdot (\underline{k} \times \underline{A}) &= (2)(-3) + (5)(2) = 4 \\
\underline{D} \cdot (\underline{k} \times \underline{A}) &= (6)(-3) + (0)(2) = -18 \\
\underline{A} \cdot (\underline{k} \times \underline{C}) &= (2)(-5) + (3)(2) = -4 \\
\underline{D} \cdot (\underline{k} \times \underline{C}) &= (6)(-5) + (0)(2) = -30
\end{aligned}
\qquad (33)
$$

Substituting equation (33) into equation (13), we obtain

$$
\begin{aligned}
a &= 12 - 19\cos\theta_j - 4\sin\theta_j \\
b &= -30 + 4\cos\theta_j - 19\sin\theta_j \\
c &= 19 - 12\cos\theta_j + 18\sin\theta_j
\end{aligned}
\qquad (34)
$$

We can now compute the values of ϕ_j at the various rotations of θ_j by substituting equation (34) into equation (12). These values are shown in Table 1. Note that the output link may assume two positions defined by ϕ_{ja} and ϕ_{jb}.

Equation (21) is now used to calculate the angular velocity of the output link. The results of $\dot{\phi}_j$ are also shown in Table 1 for the different input angles θ_j.

Since $\ddot{\theta}_j = 0$, equation (24) becomes

$$\ddot{\phi}_j = \frac{-\dot{\underline{B}}_j \cdot \dot{\underline{B}}_j - (\underline{A}_j \cdot \underline{B})\dot{\theta}_j^2 + (\underline{C}_j \cdot \underline{B}_j)\dot{\phi}_j^2}{(\underline{C}_j' \cdot \underline{B}_j)} \qquad (35)$$

Using the obtained values of ϕ_j and $\dot{\phi}_j$, we can now obtain $\ddot{\phi}_j$ using equation (35). These are all shown in Table 1.

From the results for ϕ_{ja} and ϕ_{jb}, we can recognize that the "correct" or actual solutions are ϕ_{jb}, $\dot{\phi}_{jb}$ and $\ddot{\phi}_{jb}$. We can deduce this from the fact that at $\theta_j = 360$, the solution is $\phi_{jb} = 0$ which is to be expected. Also, for the initial increment of 20°, the smaller ϕ_{jb} is the more realistic value.

Table 1. Results of Numerical Example

θ_j	ϕ_{ja}	ϕ_{jb}	$\dot{\phi}_{ja}$	$\dot{\phi}_{jb}$	$\ddot{\phi}_{ja}$	$\ddot{\phi}_{jb}$
20	143.12	12.02	−0.24	0.65	−0.47	−0.17
40	139.77	25.34	−0.10	0.68	−0.33	−0.02
60	138.75	38.78	0.00	0.66	−0.29	0.10
80	139,79	51.54	0.10	0.61	−0.31	0.20
100	143.00	62.92	0.22	0.52	−0.36	0.31
120	148.69	72.15	0.35	0.40	−0.38	0.37
140	157.02	78.81	0.8	0.27	−0.34	0.37
160	167.68	82.91	0.58	0.14	−0.24	0.33
180	180.01	180.01	0.65	0.65	−0.14	0.14
200	−166.73	84.47	0.68	−0.06	−0.03	0.30
220	−153.26	82.05	0.66	−0.18	0.11	0.40
240	−140.58	76.76	0.59	−0.36	0.34	0.67
260	−130.44	66.62	0.39	−0.69	0.90	1.29
280	−127.15	47.23	−0.15	−1.30	2.46	2.09
300	−141.19	15.81	−1.32	−1.63	3.30	−1.41
320	−173.08	−6.81	−1.55	−0.52	−1.69	−3.31
340	162.75	−8.37	−0.88	0.25	−1.64	−1.31
360	149.87	0.00	−0.46	0.54	−0.84	−0.48

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