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Soil Incompressibility by 'Mixed' and Penalty Methods

by

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ABSTRACT

In this paper two numerical methods of treating soil incompressibility are discussed, namely: (1) solution by 'mixed' formulation and (2) solution by penalty formulation. The numerical methods presented herein are finite element-based and assume that in the no-flow (undrained) condition, the deformation and pore pressure behavior of a saturated soil medium can be analyzed either by considering a two-phase soil-water relationship or by a single-phase continuum formulation.

INTRODUCTION

When a saturated soil mass deforms at a constant water content (or void ratio), the accompanying volume change is nil and the soil mass is considered to be incompressible. Soil incompressibilities occur in the undrained condition where no flow of pore fluid is involved.

In this paper two methods of numerically treating soil incompressibilities in the finite element framework are discussed, namely: (1) solution by 'mixed' formulation in which the displacement and pore pressure degrees of freedom are coupled by a virtual work or variational equation, and (2) solution by penalty formulation in which the incompressibility constraint is incorporated by lumping the bulk stiffness of water with that of the soil skeleton. These two methods correspond, respectively, to conditions in which the behavior of the saturated soil medium is characterized using a two-phase soil-water description and a single-phase soil-water continuum formulation.

Inherent in both methods is a numerical problem called *mesh locking,* a problem in which the volumetric stiffness of the material tends to dominate the numerical solution. In the 'mixed' method, the problem of mesh locking is avoided by a judicious choice of good finite elements which have been shown to exhibit compatible interpolations of displacement and pore pressure fields. In the penalty method, the same problem is treated using the selective reduced integration approach on the strain-displacement matrix $\underset{\sim}{B}$ as proposed by Hughes [6, 7].

SOLUTION BY 'MIXED' FORMULATION

A two-phase water-soil structure formulation involves explicit segregation of effective stresses and pore pressures in a fully saturated soil mass. By employing the virtual work or variational principle, a matrix equation coupling the unknown nodal displacements and nodal pore pressures can be obtained.

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Review of Previous Work

A finite element formulation for a general three-dimensional consolidation problem using the variational concepts was presented by Borja [2,3]. An incremental matrix equation during the period of transient pore pressure dissipation was derived, coupling the unknown nodal displacement increments $\Delta \underset{\sim}{d}$ and nodal pore pressure increments $\Delta \underset{\sim}{p}$, viz:

$$\begin{bmatrix} \underset{\sim}{\bar{K}} & \underset{\sim}{G} \\ \underset{\sim}{G}^T & \beta\Delta t \cdot \underset{\sim}{M} \end{bmatrix} \begin{Bmatrix} \Delta \underset{\sim}{d} \\ \Delta \underset{\sim}{p} \end{Bmatrix} = \begin{Bmatrix} \Delta \underset{\sim}{F} \\ \Delta \underset{\sim}{H} - \Delta t \cdot \underset{\sim}{M} \cdot \underset{\sim}{p}_n \end{Bmatrix} \tag{1}$$

where $0 \leqslant \beta \leqslant 1$ is the time-integration parameter, Δt is the time step such that $t_{n+1} = t_n + \Delta t$, $\underset{\sim}{p}_n$ are the known nodal pore pressures at (current) time instant t_n, and the global matrices $\underset{\sim}{\bar{K}}$, $\underset{\sim}{G}$, $\underset{\sim}{G}^T$, $\underset{\sim}{M}$, and $\{\Delta \underset{\sim}{F}, \Delta \underset{\sim}{H}\}$ are the average tangent stiffness matrix, coupling matrices, flux matrix, and force vectors, respectively, whose explicit definitions will be given subsequently.

The above matrix equation (1) has been shown to work in one- and two-dimensional plane strain and axi-symmetric (torsionless) applications on elastostatic and elasto-plastic strain-hardening materials [3].

The Condition of Incompressibility

The undrained condition may be analyzed using (1) by assuming that the saturated soil medium is loaded "instantaneously," or quickly enough to prevent drainage. The displacements can, therefore, be considered as volume change-free and the resulting strains become purely distortional.

Numerically, such a condition can be achieved [5] by setting Δt equal to zero in (1) to give

$$\begin{bmatrix} \underset{\sim}{\bar{K}} & \underset{\sim}{G} \\ \underset{\sim}{G}^T & \underset{\sim}{0} \end{bmatrix} \begin{Bmatrix} \Delta \underset{\sim}{d} \\ \Delta \underset{\sim}{p} \end{Bmatrix} = \begin{Bmatrix} \Delta \underset{\sim}{F} \\ \Delta \underset{\sim}{H} \end{Bmatrix} \tag{2}$$

The global matrices $\underset{\sim}{\bar{K}}$, $\underset{\sim}{G}$, $\underset{\sim}{G}^T$, $\underset{\sim}{M}$, $\Delta \underset{\sim}{F}$, and $\Delta \underset{\sim}{H}$ are then obtained from the assembly of the following finite element contributions:

1. element tangent stiffness

$$\underset{\sim}{\bar{K}}^e = \int_{\Omega^e} \underset{\sim}{B}^T \cdot \underset{\sim}{C} \cdot \underset{\sim}{B} \, d\Omega \tag{3}$$

2. element coupling matrices

$$\underset{\sim}{G}^e = -\int_{\Omega^e} \underset{\sim}{b} \cdot \underset{\sim}{\hat{N}}^T \, d\Omega \tag{4}$$

$$(\underset{\sim}{G}^e)^T = -\int_{\Omega^e} \underset{\sim}{\hat{N}} \cdot \underset{\sim}{b}^T \, d\Omega \tag{5}$$

3. element force vectors

$$\Delta \underset{\sim}{F}^e = \int_{\Omega^e} \underset{\sim}{N}^T \cdot \Delta \underset{\sim}{f} \, d\Omega \; + \int_{\Upsilon^e_h} \underset{\sim}{N}^T \cdot \Delta \underset{\sim}{h} \, d\Upsilon \; - (\underset{\sim}{K}^e \cdot \Delta \underset{\sim}{d}_g - \underset{\sim}{G}^e \cdot \Delta \underset{\sim}{p}_r) \qquad (6)$$

$$\Delta \underset{\sim}{H}^e = \int_{\Upsilon^e_s} \underset{\sim}{\hat{N}}^T \cdot \Delta s \, d\Upsilon \; + (\underset{\sim}{G}^e)^T \cdot \Delta \underset{\sim}{d}_g \qquad (7)$$

where $\underset{\sim}{B}$ is the strain-displacement matrix (section on Selective Reduced Integration on page 61); $\underset{\sim}{C}$ is the effective stress-based material stress-strain matrix; $\underset{\sim}{b}$ is the volumetric strain-displacement vector (i.e., $\underset{\sim}{b}_a^T = \{1,1,1,0,0,0\} \cdot \underset{\sim}{B}_a$); $\underset{\sim}{N}$ and $\underset{\sim}{\hat{N}}$ are collections of displacement and pore pressure shape functions, respectively; $\Delta \underset{\sim}{f}$ is the incremental body force vector; $\Delta \underset{\sim}{h}$ is the incremental prescribed traction vector; $\Delta \underset{\sim}{d}_g$ is the vector of prescribed displacements; $\Delta \underset{\sim}{p}_r$ is the vector of incremental prescribed nodal pore pressures; Δs is the incremental prescribed velocity flux, Υ^e_h is the element boundary along which traction h is prescribed; Υ^e_s is the element boundary along which flux s is prescribed; and Ω^e is the element domain. For a succinct description of the development of (2), consult Borja [3] and Small, et. al. [8].

By expressing equation (2) in incremental form, it also becomes applicable in the analysis of plasticity-related boundary-value problems by a quasi-static approach. The resulting system of simultaneous equations can then be solved by any acceptably efficient numerical technique, say by Crout elimination [9].

Choice of Finite Elements

Without loss of generality, only ('mixed') quadrilateral elements will be considered herein as candidate finite elements. Triangular elements can be obtained from their quadrilateral counterparts as two or more quadrilateral element nodes degenerate to a single point. These quadrilaterals will be further classified according to whether or not the pore pressure field is continuously interpolated on the global level. These candidate finite elements will henceforth be called continuous pressure and discontinuous pressure elements, respectively.

1. *Continuous pressure elements.* Three quadrilateral elements exhibiting a continuous pore pressure interpolation are shown in Figure 1. The numbers shown in each element node describe the nodal numbering scheme for data processing purposes. For convenience and to avoid the proliferation of nodes, the same displacement nodes will be used as pore pressure nodes.

Element 1 interpolates the displacement and pore pressure fields bilinearly. Element 2 employs the same bilinear pore pressure interpolation, but uses an eight-node serendipity interpolation of displacements. Element 3 is similar to element 2 except that the displacement interpolation is a nine-node (biquadratic) Lagrangian.

In general, the same interpolation for both displacement and pore pressure (e.g., element 1) should not be used in undrained cases or if step loadings are to be used [5]. A dire consequence of such a combination is the so-called *mesh locking,* a numerical problem that arises when the incompressibility constraints are too many. A heuristic approach called *constraint counts* was proposed by Hughes [7] to establish the ability of an element to perform well in incompressible and nearly incompressible cases.

Elements 2 and 3 employ a displacement interpolation which is one level higher than the pore pressure interpolation. Compatibility consideration suggests that both pore pressure and stress (or strain) will have the same level of interpolation, which probably explains why elements 2 and 3 perform better than element 1.

Apparently, element 3 (biquadratic Lagrangian on displacements) causes the size of the matrix problem to increase by n_{sd} (= number of spatial dimensions), the number of additional displacement components, for each element considered. However, the additional displacement degrees of freedom corresponding to the central node for this element can be eliminated on the element level prior to global assembly (i.e., static condensation by Gaussian elimination). Possible advantages of element 3 over element 2 are the convenience in isoparametrically generating nodal data/floating point information during the input phase and that the convergence characteristics of this element have been previously established [7].

In practice, a continuous pore pressure interpolation makes it possible to incorporate pore pressure-related boundary conditions such as prescribed pressures $\tilde{p}_r$ and nodal flux rates s in the matrix equation because the displacement and pore pressure element boundaries can be made to coincide. Continuous pressure elements also provide a smooth numerical transition from an undrained analysis to that of a transient pore pressure diffusion (consolidation) because the boundary conditions in the consolidation phase can be matched by the pressures on the same boundaries at the end of the undrained phase.

2. *Discontinuous pressure elements.* Undrained problems do not usually require explicit boundary conditions on pore pressures, nor does the variational formulation leading to matrix equation (2) include pore pressure gradients [3]. Thus the pore pressure shape functions may be discontinuous across element boundaries. This relaxation in continuity is computationally advantageous because the pore pressure degrees of freedom can be eliminated (and later recovered) on the element level prior to global assembly, allowing the global equations to be structured without these unknown pressures (again, by static condensation).

Figure 2 shows two typical discontinuous pressure elements. Element 4 contains a central pore pressure node representing a constant pore pressure interpolation within the element domain. Element 5 is a three-pore pressure-node element whose convergence characteristics have been previously tested [7].

It can be seen that by relaxing the continuity of pore pressures, a wider range of interpolations becomes available for pressures than for displacements, offering more possible combinations of displacement and pore pressure ('mixed') elements to choose from. As in continuous pressure elements, however, not all possible combinations of displacement and pore pressure interpolations work because of the mesh locking effects.

Remarks.

In practice, the global coefficient matrix is assembled to contain only the upper triangular elements, on account of symmetry, that are enveloped by the "profile" or "skyline." The elements of this matrix can be stored in a vector while keeping track of the addresses of the diagonal elements. During the factorization process, the zero elements along the diagonal become nonzero if the displacement and pore pressure nodes are properly numbered [5]. If node numbering is not properly ordered, it is possible that a zero diagonal element will be encountered during the factorization process, necessitating a node renumbering. For this reason, a simultaneous equation solver with pivoting is desirable.

Equation (2) shows that the displacement and pore pressure degrees of freedom are explicitly segregated, causing the matrix of coefficients to have a large bandwidth. These degrees of freedom, however, may be interspersed (i.e., arrange all unknowns for each node next to each other) making the coefficient matrix more closely banded.

SOLUTION BY PENALTY FORMULATION

Since no flow of pore fluid is involved in undrained problems, numerical analysis can also be carried out using a single-phase soil-water continuum formulation. Let the bulk stiffness of water be λ_w, the stress-strain matrix $\bar{\underset{\sim}{C}}$ for the soil mass is given by

$$\bar{\underset{\sim}{C}} = \underset{\sim}{C} + \underset{\sim}{C}_w \tag{8}$$

where $\underset{\sim}{C}$ is the effective stress-based stress-strain matrix (see section on the Condition of Incompressibility on page 57) and

$$\underset{\sim}{C}_w = \lambda_w \cdot \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}, \quad \underset{\sim}{1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{9}$$

The resulting matrix equation is obtained from (2) by suppressing the pore pressure degree of freedom, viz:

$$\bar{\underset{\sim}{K}} \cdot \Delta \underset{\sim}{d} = \Delta \underset{\sim}{F} \tag{10}$$

where $\underset{\sim}{K}$ is the tangent stiffness matrix obtained by assembling the finite element contributions (cf. equation (3))

$$\underset{\sim}{K}^e = \int_{\Omega^e} \underset{\sim}{B}^T \cdot \bar{\underset{\sim}{C}} \cdot \underset{\sim}{B} \, d\Omega \tag{11}$$

and $\Delta\underset{\sim}{F}$ is the global force vector obtained from the contributions $\Delta\underset{\sim}{F}^e$ given by (6).

This approach is equivalent to introducing a volumetrically stiff elastic spring represented by a large but finite λ_w, thus forcing the soil mass to become nearly incompressible. The actual bulk stiffness λ_w is not relevant; this parameter is artificially selected to be large enough so that incompressibility errors are small, yet not too large to cause numerical problems. Hughes [6] suggested that, with computer floating-point words of length 60-64 bits, the ratio λ_w/μ_{soil} (where μ_{soil} = shear modulus of the soil skeleton) may be effectively taken in the range

$$10^7 \leqslant \lambda_w/\mu_{soil} \leqslant 10^9 . \tag{12}$$

Slight compressibilities do not make the problem of mesh-locking go away. A numerical approach to this problem is by adopting a reduced numerical integration rule, a concept probably based on the presumption that errors in numerical integration compensate appropriately for the overestimation of structural stiffness due to finite element discretization [1].

A simple approach of uniform reduced integration, however, is a dangerous scheme which could reduce the rank of the stiffness matrix and cause it to become singular. An alternative approach is to use a *selective reduced integration procedure,* specifically the $\bar{\tilde{B}}$-method proposed by Hughes [6], which effectively sifts out the volumetrically stiff part of the stiffness matrix, and thus alleviates locking.

Selective Reduced Integration

Figure 3 shows three possible "no pressure" quadrilateral elements. Element 6 is a four-node quadrilateral bilinearly interpolating the displacement field. Element 7 is an eight-node serendipity, while element 8 is a nine-node (biquadratic) Lagrangian.

Figure 4 shows the normal and reduced orders of Gauss numerical integration for the isoparametric (two-dimensional) elements of Figure 3. A similar presentation may be made for isoparametric bricks in three-dimensions.

In three-dimensional problems, the strain-displacement $\tilde{B}_a$ is given by

$$\tilde{B}_a = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \\ B_2 & B_1 & 0 \\ 0 & B_3 & B_2 \\ B_3 & 0 & B_1 \end{bmatrix}, \quad B_i = \frac{\partial N_a}{\partial x_i} \cdot \tag{13}$$

Employing the selective reduced integration technique, matrix $\tilde{B}_a$ is decomposed into a volumetric part $\tilde{B}_a^{vol}$ and a deviatoric part $\tilde{B}_a^{dev}$, where

$$\tilde{B}_a^{vol} = \frac{1}{3}\begin{bmatrix} B_1 & B_2 & B_3 \\ B_1 & B_2 & B_3 \\ B_1 & B_2 & B_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

and

$$\tilde{B}_a^{dev} = \tilde{B}_a - \tilde{B}_a^{vol} \tag{15}$$

The volumetric part $\tilde{B}_a^{vol}$ is replaced by an 'improved' volumetric contribution denoted by $\bar{\tilde{B}}_a^{vol}$, i.e.,

$$\bar{\tilde{B}}_a^{vol} = \begin{bmatrix} \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{16}$$

where the $\bar{B}_i$'s are defined by the expression

$$\bar{B}_i(\tilde{\xi}) = \sum_{\ell=1}^{\bar{n}_{int}} \bar{N}_\ell(\tilde{\xi}) \cdot B_{i\ell} \tag{17}$$

in which $\underset{\sim}{\xi}$ are the element natural coordinates, $\bar{n}_{int}$ is the number of integration points in the reduced rule, and the $\bar{N}_\ell$'s are special sets of shape functions associated with the locations $\bar{\underset{\sim}{\xi}}_\ell$ of Gauss integration points in the reduced rule, i.e., $\bar{N}_k(\bar{\underset{\sim}{\xi}}_\ell) = \delta_{kl}$ (δ_{kl}= Kronecker delta) for $1 \leqslant k,\ l \leqslant \bar{n}_{int}$.

Selective integration is obtained by taking

$$B_{i\ell} = B_i(\bar{\underset{\sim}{\xi}}_\ell) \cdot \tag{18}$$

Figure 4 summarizes this scheme for various quadrilateral elements considered. This procedure may be applied to any arbitrary anisotropic and/or nonlinear situation, and to axisymmetric or rectilinear configurations [6].

Interpretation of results

Upon evaluation of $\Delta \underset{\sim}{d}$ from (10), the volumetric strain ϵ_v^e for the eth element can be interpolated within the element domain Ω^e from the formula

$$\epsilon_v^e = \underset{\sim}{b}^T \cdot \Delta \underset{\sim}{d}^e \tag{19}$$

where $\Delta \underset{\sim}{d}^e$ is the vector $\Delta \underset{\sim}{d}$ localized on the element level and $\underset{\sim}{b}$ is the volumetric strain-displacement vector of Section on the Condition of Incompressibility. The strain given by (19) is a measure of the element's degree of compressibility, and may be driven down to a value close to zero by choosing a large but finite λ_w.

The pore pressure generated is given by

$$p = \lambda_w \ \epsilon_v^e \tag{20}$$

while the effective stresses are interpolated from the expression

$$\underset{\sim}{\sigma}^e = \underset{\sim}{C} \cdot \underset{\sim}{B} \cdot \Delta \underset{\sim}{d}^e \ . \tag{21}$$

Theoretically, the above pore pressure should approach its exact value in the perfectly incompressible condition as $\lambda_w \to \infty$ and $\epsilon_v^e \to 0$.

Remarks.

The method presented above has the Lagrange multiplier method as its basis, in which the penalty parameter λ_w is the Lagrange multiplier. The same method is discussed in [7] for a one-phase material by defining p as the hydrostatic stress ($=\sigma_{ii}/3$).

SUMMARY AND CONCLUSIONS

Two finite element-based numerical methods for analyzing boundary-value problems in a saturated soil medium incorporating the incompressibility condition were presented in this paper. Possible practical uses of these methods are in the analysis of an undrained type of problem where the flow of pore fluid is suppressed or in problems where the rate of loading is much faster than the rate at which the generated excess pore pressure is dissipated.

The numerical methods presented herein were based on the assumption that in the no-flow condition, the pore pressure/deformation behavior of a saturated soil medium can be analyzed either by considering a single-phase soil-water continuum or by a two-phase formulation involving explicit segregation of excess pore pressure and displacement degrees of freedom coupled by a virtual work or variational equation. The numerical problem of mesh locking was treated by a judicious choice of finite elements that work and/or employing a non-standard numerical method, so-called selective reduced integration on the strain-displacement matrix $\underset{\sim}{B}$ [6].

While the methods presented in this paper deal solely with problems involving incompressibility, provisions can be made, particularly on the 'mixed' method, to accommodate a diffusion type of problem (consolidation) over and beyond the undrained phase should the need to numerically simulate an undrained-consolidation sequence of loading arises.

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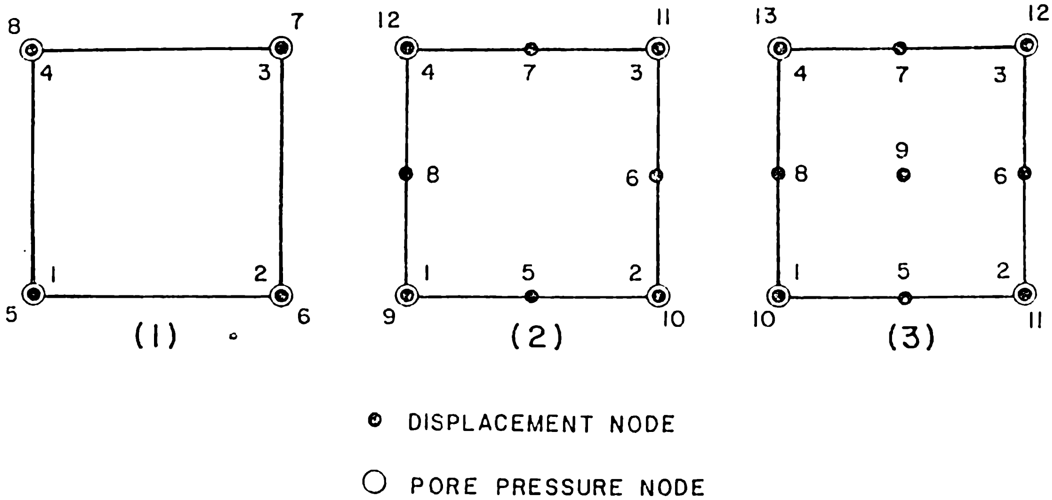


Figure 1. Continuous Pressure Elements

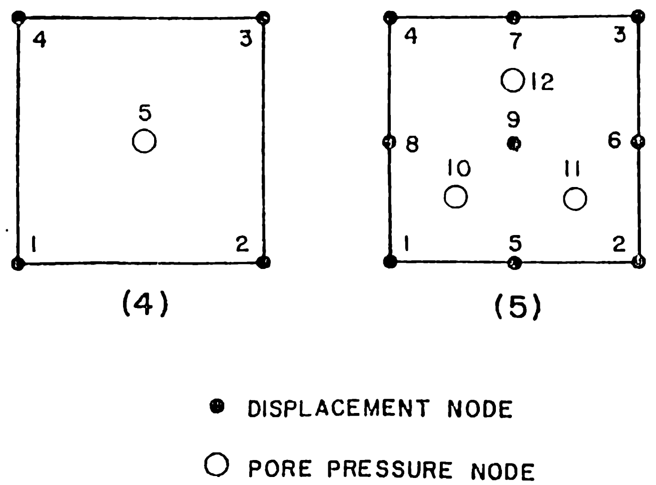


Figure 2. Discontinuous Pressure Elements

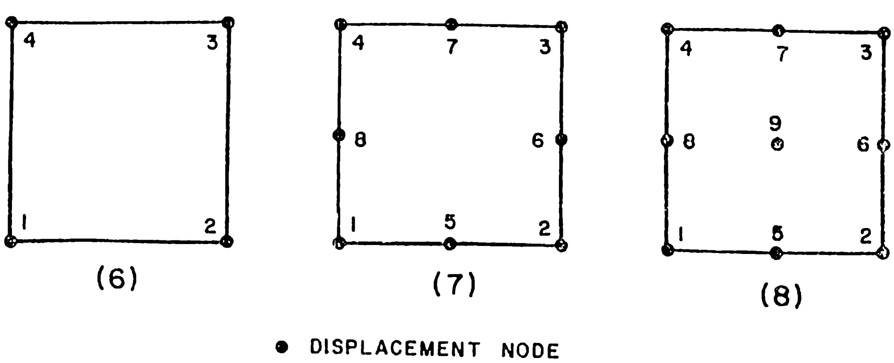


Figure 3. Standard Quadrilateral Elements

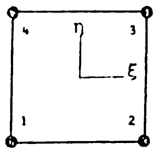
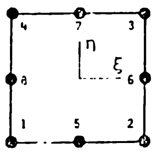
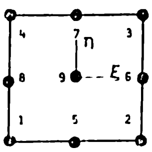
ELEMENT	NORMAL RULE	REDUCED RULE
	2x2 QUADRATURE $n_{int} = 4$ $(\xi,\eta)_\ell = (+a,+a)_1\ ,\ (+a,-a)_2$ $(-a,+a)_3\ ,\ (-a,-a)_4$ $a = 1/\sqrt{3}$	1-PT. QUADRATURE $\bar{n}_{int} = 1$ $(\bar{\xi},\bar{\eta})_\ell = (0,0)_1$ $\bar{N}_1 = 1$
 	3x3 QUADRATURE $n_{int} = 9$ $(\xi,\eta)_\ell = (+b,+b)_1\ ,\ (+b,\ 0)_2$ $(+b,-b)_3\ ,\ (\ 0,+b)_4$ $(\ 0,\ 0)_5\ ,\ (\ 0,-b)_6$ $(-b,+b)_7\ ,\ (-b,\ 0)_8$ $(-b,-b)_9\ ,\ b = \sqrt{3/5}$	2x2 QUADRATURE $\bar{n}_{int} = 4$ $(\bar{\xi},\bar{\eta})_\ell = (+a,+a)_1\ ,\ (+a,-a)_2$ $(-a,+a)_3\ ,\ (-a,-a)_4$ where $a = 1/\sqrt{3}$ $\bar{N}_1 = c(a+\xi)(a+\eta)$ $\bar{N}_2 = c(a+\xi)(a-\eta)$ $\bar{N}_3 = c(a-\xi)(a+\eta)\ ;\ c = \frac{3}{4}$ $\bar{N}_4 = c(a-\xi)(a-\eta)$

Figure 4. Comparison Between Normal Quadrature Rule and Selective Reduced Integration Rule