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A Unified Method for MSP Synthesis of Planar Four-Bar Function Generators

by

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ABSTRACT

A review of the synthesis of planar four-bar function generators results in a more unified approach to the problem. The question of number of parameters available for synthesis vis-a-vis the number of synthesis equations is reconciled. Furthermore, the parameters used for synthesis are simply the coordinates of the vectors that define the mechanism. The method developed gives very explicit equations in these parameters and with slight modifications and derivations become applicable for higher order synthesis. Thus, the method is generalized for multiply-separated-position (MSP) synthesis of planar 4-bar function generators.

INTRODUCTION

The milestone work of Freudenstein (1) on the approximate synthesis of four-bars has been the basis for a lot of activity and research in the general subject of synthesis of mechanisms. One of its most significant contributions has been the so-called "Freudenstein Equations" which have been used and accepted as the tool for synthesizing planar four-bar function generators. These equations have been modified to accommodate other motion requirements like pressure angles and ranges of motion.

There are, however, some difficulties with the Freudenstein method when using it to synthesize for increasing number of positions. That is, it appears that one has to solve a system of non-linear equations with more equations than unknowns. There is nothing inconsistent, however, since one has to include some auxiliary equations that must also be satisfied. The difficulty really lies in the fact that these equations/unknowns are not obvious and explicit in nature. In some cases, they come about during the solution process. One needs only to examine references (2) and (3) to appreciate the problem. In these references, the solutions have been shown for some of the synthesis cases.

These difficulties were again encountered during a graduate course in Mechanisms at the Department of Mechanical Engineering. This incident motivated the author to devise, if not develop, a more coherent or unified approach to the problem.

After reviewing the basis and assumptions of Freudenstein, it was found out that only by changing these basic assumptions can one develop the synthesis equations that are not of the same form as the Freudenstein equations. In short, the Freudenstein equations are optimum for the problem with its particular set of assumptions.

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Having changed the basis, initial attempts by the author have resulted in synthesis equations that – although were coherent in terms of number of unknowns and equations – were quite cumbersome and highly non-linear even for very simple cases.

THE UNIFIED METHOD

The unified method can be best described as a reconciled approach in terms of number of equations and number of unknowns. At the very start, the parameters for synthesis are identified and as successive positions are required for synthesis, so are corresponding number of parameters made available for synthesis. In this method of synthesis, the mechanism is defined at its initial position by the j counter set to 1. A synthesis equation (that is, one) is obtained when the mechanism is made to satisfy an additional position different from the initial position. Thus, 2 position synthesis ($j = 2$) means that one synthesis equation is obtained and only one parameter can be solved for – the rest have to be specified; 3 position means two synthesis equations and so on. This method also makes higher order synthesis possible thru the use of the higher order form of the equation and still with the idea of having an equal number of equations and unknowns. Things are still consistent because a position is counted whether it is a higher order (infinitesimally separated) position or not.

DEVELOPMENT OF THE SYNTHESIS METHOD

The new method of synthesizing four-bar function generators is based on the following steps:

a. Redefine the parametric definition of the four-bar function generator.
b. Develop the equation of motion based on *displacements* in vector form.
c. Reformulate the equation of motion to be functions of the parameters defining the four-bar function generator.
d. Extend the applicability of the synthesis equations for higher order synthesis.
e. Solve the different cases of different numbers of positions, specified and available parameters for synthesis.

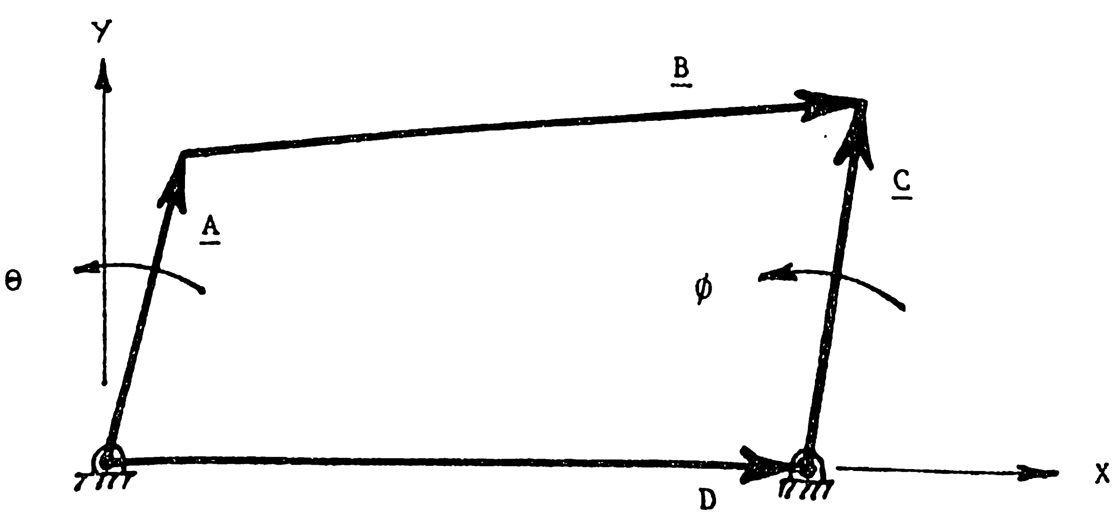

Figure 1. The Four-Bar and its Vector Representation

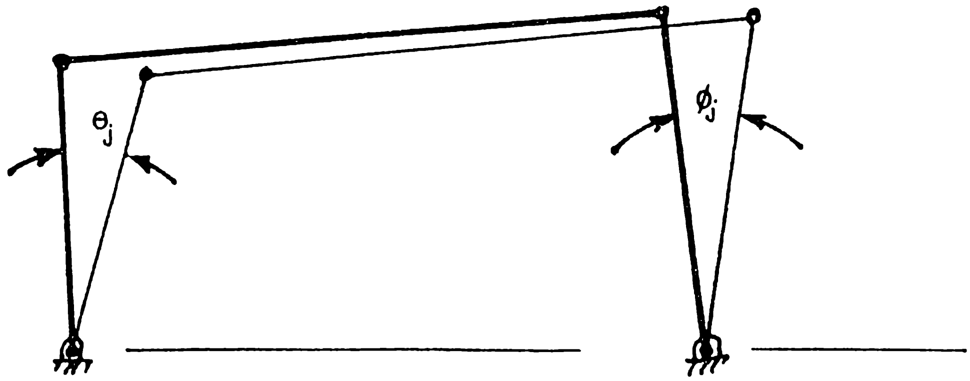

Figure 2. The Four-Bar at its Displaced Position

Figure 1 shows the four-bar as defined by vectors $\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{D}$, and by the motion parameters ϕ and θ. For function generation, the vector $\underline{D}$ is made into a unit vector in the X direction to fix the scale of the mechanism to be synthesized. It should be noted that a four-bar that is scaled-up or down will still have the same relations in terms of its motion parameters ϕ and θ. Thus, we only have two vectors that are available for synthesis. They are any two from the set of $\underline{A}$, $\underline{B}$, and $\underline{C}$. From previous attempts at this task of synthesizing, it was found out that the combination of $\underline{A}$ and $\underline{C}$ as the vectors to be synthesized for is most convenient.

Referring to Figure 2 which shows the displaced position of the four-bar, the equation of motion as derived in reference (4) is

$$\begin{aligned}&\cos\phi_j\,[(\underline{C}\cdot\underline{D}) - (\underline{A}\cdot\underline{C})\cos\theta_j - \underline{C}\cdot(\underline{k}\times\underline{A})\sin\theta_j] + \\ &\sin\phi_j\,[(\underline{k}\times\underline{C})\cdot\underline{D} - (\underline{k}\times\underline{C})\cdot\underline{A}\cos\theta_j - (\underline{A}\cdot\underline{C})\sin\theta_j] + \\ &[(\underline{C}\cdot\underline{C} + \underline{D}\cdot\underline{D} + \underline{A}\cdot\underline{A} - \underline{B}\cdot\underline{B})/2 - (\underline{A}\cdot\underline{D})\cos\theta_j - \underline{D}\cdot(\underline{k}\times\underline{A})\sin\theta_j] = 0\end{aligned} \qquad (1)$$

where $\underline{k}$ is the unit vector along the Z axis.

To eliminate the vector $\underline{B}$ which is really dependent on $\underline{A}$, $\underline{C}$ and $\underline{D}$, the loop closure equation is written as

$$\underline{B} = \underline{C} + \underline{D} - \underline{A} \qquad (2)$$

Taking the dot product with itself, we get

$$\underline{B}\cdot\underline{B} = \underline{A}\cdot\underline{A} + \underline{C}\cdot\underline{C} + \underline{D}\cdot\underline{D} - 2\underline{A}\cdot\underline{C} - 2\underline{A}\cdot\underline{D} + 2\underline{C}\cdot\underline{D} \qquad (3)$$

Substituting equation (3) into equation (1) will give us

$$\begin{aligned}&\cos\phi_j(\underline{C}\cdot\underline{D}) - \cos\phi_j\cos\theta_j(\underline{A}\cdot\underline{C}) - \cos\phi_j\sin\theta_j(\underline{k}\times\underline{A})\cdot\underline{C} \\ &+ \sin\phi_j(\underline{k}\times\underline{C})\cdot\underline{D} - \sin\phi_j\cos\theta_j(\underline{k}\times\underline{C})\cdot\underline{A} - \sin\phi_j\sin\theta_j(\underline{A}\cdot\underline{C}) \\ &+ \underline{A}\cdot\underline{C} + \underline{A}\cdot\underline{D} - \underline{C}\cdot\underline{D} - \cos\theta_j(\underline{A}\cdot\underline{D}) - \sin\theta_j(\underline{k}\times\underline{A})\cdot\underline{D} = 0\end{aligned} \qquad (4)$$

We now write the *design* vectors as

$$\underline{A} = a_1\underline{i} + a_2\underline{j} \tag{5}$$

and $$\underline{C} = c_1\underline{i} + c_2\underline{j} \tag{6}$$

These are our design vectors because they define the unique four-bar that will satisfy the motion requirements. In essence now, the procedure is to specify the motion requirements as ϕ_j and θ_j and try to solve for a_1, a_2, c_1 and c_2. Of course, depending on the number of positions, (in turn, the number of equations), there can only be a set number of unknowns in a_1, a_2 c_1 and c_2.

Without going through the lengthy manipulation, equation (6) when substituted into equation (4) will result in

$$\begin{aligned}&(1-\cos\theta_j)a_1 + \sin\theta_j a_2 + (\cos\phi_j - 1)c_1 - \sin\phi_j c_2 + \\ &(1-\cos\phi_j\cos\theta_j - \sin\phi_j\sin\theta_j)a_1c_1 + (\sin\phi_j\cos\theta_j - \cos\phi_j\sin\theta_j)a_1c_2 + \\ &(\cos\phi_j\sin\theta_j - \sin\phi_j\cos\theta_j)a_2c_1 + (1-\cos\phi_j\cos\theta_j - \sin\phi_j\sin\theta_j)a_2c_2 = 0\end{aligned} \tag{7}$$

Equation (7) is now the synthesis equation for any number from j = 2 to j = 2, 3, 4, 5.

Thus, we can write equation (7) as

$$\begin{aligned}&K_{1j}a_1 + K_{2j}a_2 + K_{3j}c_1 + K_{4j}c_2 + \\ &K_{5j}a_1c_1 + K_{6j}a_1c_2 + K_{7j}a_2c_1 + K_{8j}a_2c_2 = 0 \\ &\qquad j = 2,\ 3,\ \text{up to } 5\end{aligned} \tag{8}$$

The synthesis equation as given by equation (8) is really the displacement synthesis equation since the K_{ij}'s are just functions of ϕ_j's and θ_j's. If we now take the time derivatives of equation (8), the form of the equation will not change with respect to the design variables a_1, a_2, c_1 and c_2. Thus, the generalized MSP synthesis equation is simply

$$\begin{aligned}&K_{ij}^{n}a_1 + K_{2j}^{n}a_2 + K_{3j}^{n}c_1 + K_{4j}^{n}c_2 + K_{5j}^{n}a_1c_1 + \\ &K_{6j}^{n}a_1c_2 + K_{7j}^{n}a_2c_1 + K_{8j}^{n}a_2c_2 = 0\end{aligned} \tag{9}$$

where j = 2, 3, up to 5

and n = 0 or 1 or 2 or 3

Note also that the K_{ij}^{n}'s are simply

$$\frac{d^n}{dt^n}(K_{ij}) = K_{ij}^{n} \tag{10}$$

The important feature of equation (9) is that finitely separated positions (FSP) or infinitesimally separated positions (ISP) or combinations of the two (MSP) can be specified with ease by using the appropriate K_{ij}^{n} terms in the equation. The counter j in the equation takes on a slightly different meaning -- it is now a label for either ISP's or FSP's. The initial position is also set as the

$j = 1$ position. For every (jth) position (except $j = 1$), equation (9) is written by computing each K_{ij}^n term from the required functional relations of $[\phi_j, \dot{\phi}_j, \ddot{\phi}_j \ldots, \theta_j, \dot{\theta}_j, \ddot{\theta}_j \ldots]$ whenever ISP's are specified at that discrete position. For example, if an ISP of second order is specified, then the displacements and velocities together with the accelerations must be specified. If P, V and A stand for position, velocity and acceleration, respectively, and a dash represents separations between discrete positions, then, only clusters of P, PV or PVA can be grouped between dashes. Groups like PA or VA or V alone are not valid. An example of a valid set of specifications is the representation of the following five MSP specifications: P-PVA-P. Note also, that the $j = 1$ position refers to the initial position where ϕ_1 and θ_j are both zero.

SOLUTIONS TO THE DIFFERENT SYNTHESIS CASES

The solutions to the different cases of number of positions and parameters available for synthesis will now be presented. It should be noted that any parameter not available for synthesis is assumed to be specified.

A. Two Positions

For this case, any one of a_1, a_2, c_1 or c_2 can be considered as the unknown. In all cases, the result is a linear equation in one unknown.

B. Three Positions

There will be two equations for this and a possibility of two cases of unknown parameters. First is when either the vector $\underline{A}$ (in a_1 and a_2) or C (in c_1 and c_2) is the unknown and second is when a parameter each from $\underline{A}$ or $\underline{C}$ are the unknowns.

For the first case, the result will be two linear equations in two unknowns as shown for example when a_1 and a_2 are the unknowns.

$$(K_{1j} + K_{5j}c_1 + K_{6j}c_2)a_1 + (K_{2j} + K_{7j}c_1 + K_{8j}c_2)a_2 + (K_{3j}c_1 + K_{4j}c_2) = 0 \qquad (11)$$

$$j = 2, 3$$

For the second case and letting a_1 and c_1 be the unknowns, we can rearrange the equation to give

$$[(K_{1j} + K_{6j}c_2) + K_{5j}c_1]a_1 + [(K_{2j} + K_{4j}c_2 + K_{8j}a_2c_2) + (K_{3j} + K_{7j}a_2)c_1] = 0 \qquad (12)$$

$$j = 2, 3$$

Equation (12) is simplified as

$$(A_j + B_jc_1)a_1 + (C_j + D_jc_1) = 0 \qquad (13)$$

$$j = 2, 3$$

The solution to this system of non-linear equation is obtained by first finding the eliminant expressed as the following determinant (please see reference (5) for further details),

$$\begin{vmatrix} (A_2 + B_2c_1) & (C_2 + D_2c_1) \\ (A_3 + B_3c_1) & (C_3 + D_3c_1) \end{vmatrix} = 0 \qquad (14)$$

This will give us a quadratic equation in c_1, the roots of which are solutions for c_1. They are then substituted in any of the two equations of (13) to get a_1. Note here that there are two possible sets of solutions.

C. Four Positions

The only possible case is that one vector in its two parameters and one parameter of the other vector are the unknowns. The solutions will be presented for the case of a_1, a_2 and c_1 as the unknowns. Any other combination of unknowns will have exactly the same form of the solution since the distribution of the variables and their products are the same.

The synthesis equations are rearranged so that the coefficients of the parameters a_1 and a_2 are functions of c_1. Thus, we get

$$[(K_{ij} + K_{6j}c_2) + (K_{5j}c_1)]a_1 + [(K_{2j} + K_{8j}c_2) + K_{7j}c_1]a_2$$
$$+ K_{4j}c_2 + K_{9j}c_1 = 0 \qquad (15)$$

Again, we can write equation (15) as

$$(A_j + B_jc_1)a_1 + (C_j + D_jc_1)a_2 + (E_j + F_jc_1) = 0 \qquad (16)$$
$$j = 2, 3, 4$$

The solution for c_1 is obtained from the determinant below set to zero

$$\begin{vmatrix} (A_2 + B_2c_1) & (C_2 + D_2c_1) & (E_2 + F_2c_1) \\ (A_3 + B_3c_1) & (C_3 + D_3c_1) & (E_3 + F_3c_1) \\ (A_4 + B_4c_1) & (C_4 + D_4c_1) & (E_4 + F_4c_1) \end{vmatrix} = 0 \qquad (17)$$

The roots of the cubic polynomial obtained from the determinant called the eliminant will now give the solutions for c_1. For every value of c_1, we then substitute it in any two equations of (15). This will give us two linear equations in two unknowns a_1 and a_2. Thus, there are three possible sets of solutions for the four position synthesis of the four-bar function generator.

D. Five Positions

For this case, the unknowns are all of the four parameter a_1, a_2, c_1 and c_2. The synthesis equations are made into the following form:

$$A_jc_1 + (B_j + C_jc_1)a_1 + (D_j + E_jc_1)a_2 + F_jc_2 + G_ja_1c_2 + H_ja_2c_2 = 0 \qquad (18)$$
$$j = 2, 3, 4, 5$$

The eliminant of this system of equations is obtained by multiplying all of the equations by c_2 twice so that a system of twelve homogeneous equations in the twelve unknowns a_1, a_2, a_1c_2, a_2c_2, c_2, 1, $a_1c_2^2$, $a_2c_2^2$, c_2^2, $a_1c_2^3$, $a_2c_2^3$ and c_2^3.

From this system, the eliminant is obtained by setting the determinant of the coefficients of these twelve unknowns to zero. This is as follows:

$$\begin{vmatrix} A_jc_1 & (B_j + C_jc_1) & (D_j + E_jc_1) & F_j & G_j & H_j & & & & & & \\ & & & A_jc_1 & (B_j + C_jc_1) & (D_j + E_jc_1) & F_j & G_j & H_j & & & \\ & & & & & & A_jc_1 & (B_j + C_jc_1) & (D_j + E_jc_1) & F_j & G_j & H_j \end{vmatrix} = 0 \qquad (19)$$

$$j = 2, 3, 4, 5$$

The resulting eliminant is a fifth degree polynomial in c_1. There is a convenient technique shown in reference (6) for evaluating polynomials from these types of determinants. The roots of this polynomial now give the solutions to c_1. With c_1 known, we now consider any three equations of (18) and rearrange the equations to give

$$(A_j' + B_j'c_2)a_1 + (C_j' + D_j'c_2)a_2 + (E_j' + F_j'c_2) = 0 \qquad (20)$$

$$j = 2, 3, 4$$

The solution for c_2, a_1 and a_2 is now obtained in a manner similar to that for the four position synthesis problem. Finally, therefore, there are theoretically 15 possible solutions to the five position synthesis problem. This is because any one value of c_1 (there are five) will have three possible solutions for c_2.

CONCLUSIONS AND RECOMMENDATION

The unified method has been shown including the solution procedures for all the different cases presented. Except for the case of the maximum number of positions (i.e., five), the solutions are quite straightforward. The solution to the five-position problem will require the assistance of a computer since there is a need to evaluate a 12 x 12 determinant.

Further study in eliminating possible branching and order problems would enhance the solution to the synthesis problem.

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