"With this formulation, the value of some forces that do no work can be determined..."

An Alternative Formulation of the Work-Energy Principle

by

Edgardo S. Pacheco, Ph.D.*

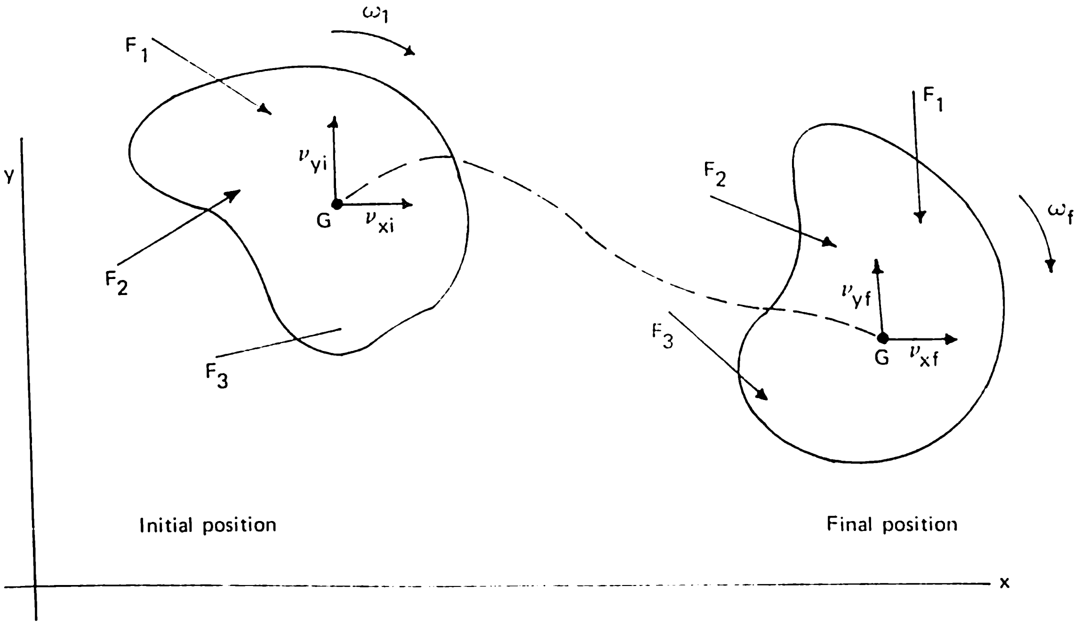
The work-energy method of solving problems in kinetics is well known to students of dynamics. It is usually the most expeditious method of solving problems in kinetics that explicitly involve displacements. The very nature of the method, however, precludes the determination of the values of forces that do no work. For example, the values of the reactions at fixed supports can not be found by this method. Also, since only a single scalar equation can be written for each body of the system under consideration, situations frequently arise when there are more unknowns than there are independent work-energy equations. In these cases, supplementary equations are obtained by using the force-mass-acceleration or impulse-momentum methods.

In this paper, the work-energy principle will be formulated in such a way that three independent equations can be written for each rigid body of a dynamical system that is in plane motion (other than translation). This formulation is based on a little known procedure for deriving the work-energy equation. The reader is referred to "Theoretical Mechanics" by W. D. MacMillan (McGraw Hill Book Co., 1936) in which two independent scalar equations were first obtained and then added together to yield the work-energy equation.

A modification of the treatment in MacMillan's book will be presented here. It is not the intent of this paper to present another way of deriving the work-energy equation. Rather, the aim is to derive a set of three independent equations that look like work-energy equations which can be utilized for solving problems in kinetics. With this formulation, the value of some forces that do no work can be determined without resorting to the force-mass-acceleration or impulse-momentum methods. Three examples are given to demonstrate the use of the equations.

Consider a rigid body that moves along the xy plane under the action of a system of coplanar forces as shown in the figure on the next page. The body is assumed symmetrical about the xy plane.

* Director, Graduate Division, U.P. College of Engineering



The following symbols are used in this paper:

x_G	=	x coordinate of the mass center G
y_G	=	y coordinate of the mass center
a_x	=	x component of the acceleration of G
a_y	=	y component of the acceleration of G
v_i	=	initial velocity of G
v_f	=	final velocity of G
v_{xi}	=	x component of v_i
v_{yi}	=	y component of v_i
v_{xf}	=	x component of v_f
v_{yf}	=	y component of v_f
α	=	angular acceleration
ω_i	=	initial angular velocity
ω_f	=	final angular velocity
Θ	=	angular position of the body
m	=	mass of the body
I_G	=	mass moment of inertia about G

The general plane motion of a rigid body is described by the following equations:

$$\Sigma F_x = ma_x = m\frac{dv_x}{dt} \tag{1}$$

$$\Sigma F_y = ma_y = m\frac{dv_y}{dt} \tag{2}$$

$$\Sigma M_G = I_G\alpha = I_G\frac{d\omega}{dt} \tag{3}$$

Multiplying both sides of Equation 1 by dx_G (the x component of an infinitesimal displacement of the mass center) gives

$$\Sigma F_x dx_G = m \frac{dv_x}{dt} dx_G = mv_x dv_x.$$

When the preceding equation is integrated from the initial to the final position of the body, the result is

$$\Sigma \int F_x dx_G = \frac{1}{2} [v^2_{xf} - v^2_{xi}] \qquad (4)$$

A similar equation is obtained by multiplying Equation 2 by dy_G and integrating:

$$\Sigma \int F_y dy_G = \frac{1}{2} [v^2_{yf} - v^2_{yi}]. \qquad (5)$$

We next multiply Equation 3 by $d\Theta$ (an infinitesimal angular displacement of the rigid body) to obtain

$$\Sigma M_G d\theta = I_G \frac{d\omega}{dt} d\theta = I_G \omega x l \omega.$$

Integration of this equation from the initial to the final position of the body results in the equation

$$\Sigma \int M_G d\theta = \frac{1}{2} I_G [\omega^2_f - \omega^2_i]. \qquad (6)$$

Equations 4, 5 and 6 constitute a system of three independent equations that can be used for studying plane motion of rigid bodies. The relationship between these equations and the work-energy equation for plane motion will now be established.

Denoting by U_{if} the work done by all the external forces acting on a rigid body as it moves from some initial position to a final position, we have the following principle of work and energy:

$$U_{if} = \frac{1}{2} I_G [\omega^2_f - \omega^2_i] + \frac{1}{2} m [v^2_f - v^2_i]. \qquad (7)$$

The right hand side of Equation 7 is the change in the kinetic energy of the body and is equal to the sum of the right hand sides of Equations 4, 5 and 6.

To obtain the work done by the external forces, each external force will be replaced by an equivalent force-couple system at the mass center. The original force system will thus be replaced by an equivalent system consisting of couples lying on the xy plane and forces that are concurrent at the mass center. The work done by the x components of the forces concurrent at the mass center is equal to the left hand side of Equation 4. The work done by the y components of the same forces is equal to the left hand side of Equation 5. The work done by the couples is equal to the left hand side of Equation 6. Hence, the total work U_{if} is equal to the sum of the left hand sides of Equations 4, 5 and 6. In other words, when Equations 4, 5 and 6 are added, the result is Equation 7.

Another way of viewing the results that have been obtained is to consider plane motion (other than translation) as a combination of the following constituent motions:

a) translation with the mass center along an arbitrarily chosen x direction,

b) translation with the mass center along the y direction which is perpendicular to the x direction, and

c) rotation about the mass center.

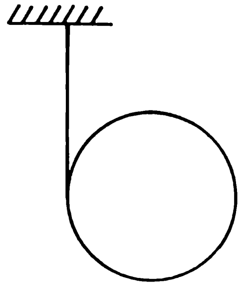
The kinetic energies during the translation along the x and y directions can be defined as $\frac{1}{2}mv_x^2$ and $\frac{1}{2}mv_y^2$, respectively. The kinetic energy during the rotational part of the motion is $\frac{1}{2}I_G\omega^2$.

The principle of work and energy for a rigid body in plane motion can now be stated as follows:

> The work done by the external forces acting on a rigid body during a constituent motion is equal to the change in the kinetic energy of the body in the same motion.

The following examples illustrate how this alternative formulation is used to determine the values of quantities not otherwise obtainable through the application of the traditional work-energy principle alone.

Example 1:

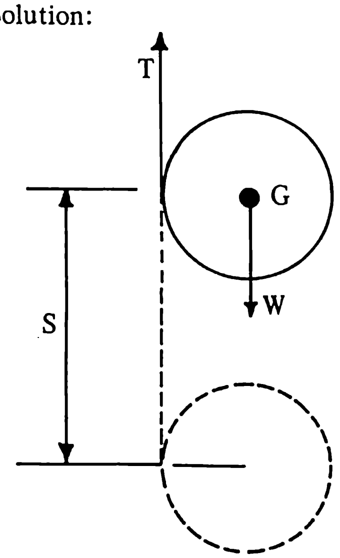


A flexible cord is wrapped around a cylinder of weight W. Find

a) the velocity of the mass center of the cylinder after it has moved a vertical distance s starting from rest.

b) the tension in the cord.

Solution:



The free body diagram of the cylinder is shown at the left. The traditional work energy equation (Eq. 7) yields

$$Ws = \frac{1}{2}\frac{W}{g}\nu_f^2 + \frac{1}{2}I_G\,\omega_f^2.$$

Since $I_G = \frac{1}{2}\frac{W}{g}r^2$ and $\omega_f = \frac{\nu_f}{r}$

where r is the radius of the cylinder, then

$$v_f = \sqrt{\frac{4gs}{3}}$$

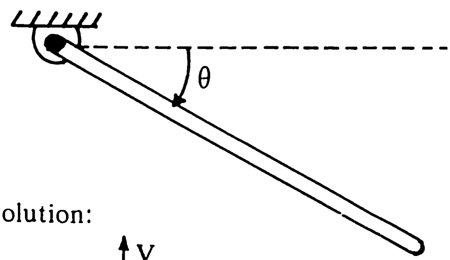
It will be noted that the work-energy equation for the complete motion does not involve T since the work done by the tension in the cord is zero. However, for a constituent motion, namely, translation in the vertical direction, the force T does work. We therefore apply Equation 5 to the cylinder and write

$$(W - T)s = \frac{1}{2}\frac{W}{g}\nu_f^2 = \frac{1}{2}\frac{W}{g}\left(\frac{4gs}{3}\right)$$

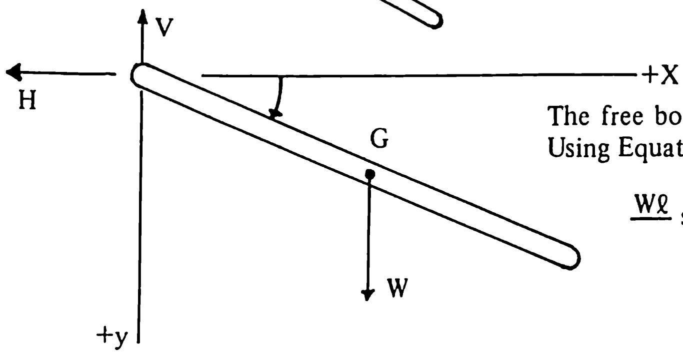
so that $T = \frac{W.}{3}$

Example 2:

A homogeneous bar of length ℓ and weight W is supported by a smooth pin at one end and is released from rest when $\Theta = 0$. Find in terms of Θ the angular velocity of the bar and the horizontal and vertical components of the reaction at the pin.



Solution:



The free body diagram of the bar is as shown. Using Equation 7, we obtain

$$\frac{W\ell}{2} \sin \theta = \frac{1}{2}\left(\frac{1}{3}\frac{W}{g}\ell^2\right)\omega_f^2$$

so that
$$\omega_f = \sqrt{\frac{3g \sin \theta}{\ell}}$$

The coordinates of the mass center G when the bar has turned through an angle Θ are

$$x_G = \frac{\ell}{2}\cos \theta \qquad y_G = \frac{\ell}{2}\sin \partial$$

Therefore, $dx_G = \frac{-\ell}{2}\sin \theta \, d\theta$ and $dy_G = \frac{\ell}{2}\cos \theta \, d\theta$

The use of Equation 4 yields

$$\int H \frac{\ell}{2}\sin \theta \, d\theta = \frac{1}{2}\frac{W}{g}\left(-\frac{\ell}{2}\sin \theta \, \omega_f\right)^2$$
$$= \frac{3W\ell}{8} \sin^3 \theta.$$

Differentiating the preceding equation with respect to Θ and solving for H, we obtain

$$H = \frac{9W}{4} \sin \theta \cos \theta.$$

The expression for V can be found by using either Equation 5 or Equation 6. Using Equation 6, we get

$$\int \frac{V\ell}{2} \cos \theta \, d\theta - \int \frac{H\ell}{2} \sin \theta \, d\theta = \frac{1}{2}\left(\frac{1}{12}\frac{W}{g}\ell^2\right)\frac{3g \sin \theta}{\ell}$$

Differentiation of this equation with respect to Θ gives

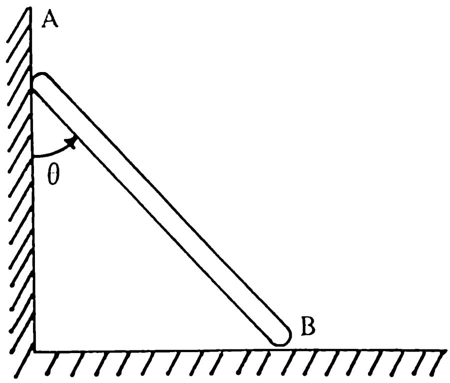
$$\frac{V\ell}{2}\cos \theta - \frac{H\ell}{2}\sin \theta = \frac{W}{8}\cos \theta.$$

Using the expression obtained for H, we get

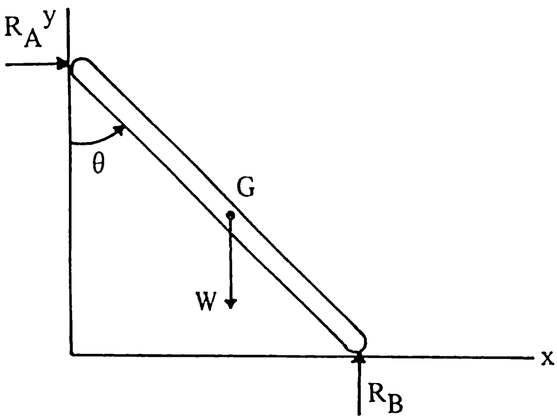
$$V = \frac{W}{4} + \frac{9W}{4}\sin^2 \theta.$$

Example 3:

The homegeneous bar AB of length ℓ and weight W is released from rest when $\Theta = 0$. Neglecting friction, find the angular velocity of the rod and the reactions at A and B in terms of Θ.



Solution:



The coordinates of the mass center of the rod are

$$x_G = \frac{\ell}{2} \sin \theta \text{ and } y_G = \frac{\ell}{2} \cos \theta.$$

Hence, $V_x = \dfrac{dx_G}{dt} = \dfrac{\ell\omega}{2} \cos \theta$

and $\nu_y = \dfrac{dy_G}{dt} = -\dfrac{\ell\omega}{2} \sin \theta$

where $\omega = \dfrac{d\theta}{dt}$.

From Equation 7,

$$W\left(\frac{\ell}{2} - \frac{\ell}{2} \cos \theta\right) = \frac{1}{2}\left(\frac{1}{12}\frac{W}{g}\ell^2\right)\omega^2$$

$$+\frac{1}{2}\frac{W}{g}\left[\left(\frac{\ell\omega}{2}\cos \theta\right)^2 + \left(-\frac{\ell\omega}{2} \sin \theta\right)^2\right]$$

so that

$$w = \sqrt{\frac{3g}{\ell}(1 - \cos \theta)} \; .$$

Equation 4 for this problem is

$$\int \frac{R_A \ell}{2} \cos \theta \, d\theta = \frac{1}{2}\frac{W}{g}\frac{\omega^2 \ell^2 \cos^2\theta}{4} = \frac{3W\ell}{8}(\cos^2 \theta - \cos^3 \theta).$$

Differentiating this equation with respect to θ, we obtain

$$\frac{R_A \ell}{2} \cos \theta = \frac{3W\ell}{8}(-2 \cos \theta . \sin \theta + 3 \cos^2 \theta \sin \theta) \cdot$$

Therefore,

$$R_A = \frac{3W}{4}(3 \cos \theta - 2) \sin \theta \; .$$

Equation 6 yields

$$\int \left(\frac{R_B \ell}{2} \sin \theta - \frac{R_A \ell}{2} \cos \theta\right) d\theta = \frac{1}{2}\left(\frac{1}{12}\frac{W}{g}\ell^2\right)\omega^2 \; .$$

Differentiating with respect to θ, we get

$$\frac{R_B \ell}{2} \sin \theta - \frac{R_A \ell}{2} \cos \theta = \frac{1}{24} \frac{W \ell^2}{g} \left(\frac{3g}{\ell} \sin \theta\right).$$

Using the previsouly obtained expression for R_A, we obtain

$$R_B = \frac{W}{4} + \frac{3W}{4} (3 \cos \theta - 2) \cos \theta.$$