"The elimination method in solving non-linear equations . . . has the advantage of revealing the maximum number of solutions possible."

Solutions to Systems of Non-Linear Equations That Arise in the Kinematic Synthesis of Spatial Mechanisms*

by

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Abstract

Determining the key dimensions of a mechanism for a prescribed performance is called the "synthesis" of the mechanism. The prescribed performance is given by specifying position or motion specifications at "precision points." These specifications lead to a system of equations whose solutions give the desired key dimensions of the mechanism. Synthesis for two or three positions generally results in a linear system of equations. Synthesis for more than three positions leads to a non-linear system. Different types of mechanims give different forms of synthesis equations. In this paper, two spatial mechanisms – a three-link mechanism with an intermediate higher pair for function generation and a five-link motion generator mechanism – are considered.

Introduction

The synthesis of spatial mechanisms has been studied by many researchers and authors. Among them were –Novodroskii [1], Levitskii and Shakvasian [2], Rao, Sandor, Kohli and Soni [3], Roth [4,5], Chen and Roth [6,7], Sandor [8], Sandor and Bisshopp [9], Suh [10,11] and Kohli and Soni [12,13], who developed various methods and analytical tools for the kinematic synthesis of spatial mechanisms.

Recently, Hernandez, Sandor and Kohli [14] proposed a three-link R-SpR**** mechanism for spatial function generation and Sandor, Kohli, Zhuang and Reinholtz [15] developed the design procedures for four-position synthesis of the RSSR-SC spatial motion generator.

The objective in the synthesis of mechanisms is to determine the key dimensions for a pre-conceived type of single input mechanism for a prescribed performance. The performance is prescribed by specifying position or motion parameters at the so-called precision points. The specified positions or motion specifications can be finitely spaced apart or they can be spaced infinitesimally close together. In the latter case, the specifications are functions of the displacement or motion derivatives. Precision requirements may be specified by finitely separated positions (FSP), infinitesimally separated positions (ISP), or combinations of ISP and FSP, called multiply-separated-position (MSP).

Kinematic synthesis of spatial mechanisms for two or three multiply-separated positions generally results in systems of linear equations. For four or more MSP synthesis, the synthesis procedure would require the solution of systems of non-linear equations.

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*Presented at the Joint National Meeting of the Institute of Management Sciences/Operations Research Society of America, April 25-27, 1983, Chicago, Ill., U.S.A.

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****Revolute, Sp: Sphere-plane, C:Cylindric, S: Spheric Pair.

Different types of mechanisms would give different forms of synthesis or design equations.

In this paper, the method used to solve the resulting non-linear systems of the R-Sp-R for six positions and the RSSR-SC for four positions will be presented.

The R-Sp-R 3-Link Spatial Function Generator

Figure 1 shows the pre-conceived 3-link revolute – sphere-plane – revolute mechanism. The minimum number of only two moving links is achieved by employing an intermediate higher pair. This mechanism is synthesized by prescribing the rotational motion of link 2 to be coordinated with the input motion of link 1. The same figure also shows the vectors that define the geometry of the mechanism at its initial position. These are:

- $\hat{u}_A$ unit vector defining the direction of the axis of link 1 at the revolute pair A. This axis is made to coincide with the X-axis such that $\hat{u}_A = \hat{i}$
- $\hat{u}_B$ unit vector defining the axis of the revolute pair B. It is defined by the skew angle θ_s measured about the common perpendicular of $\hat{u}_A$ and $\hat{u}_B$, collinear with the Y-axis. Thus, $\hat{u}_B = \cos\theta_s \hat{i} - \sin\theta_s \hat{k}$.
- Q vector along the Y-axis, the common perpendicular of $\hat{u}_A$ and $\hat{u}_B$ from A to B.
- R vector from the origin to the point R, the sphere center, a point fixed to link 1.
- R' vector from the origin to the point R', a point fixed to link 2 and initially coincident with R. Initially then, $R = R'$
- $\hat{A}$ a unit vector fixed to link 2 and perpendicular to the plane in the Sp pair.
- θ angle of rotation of link 1 about $\hat{u}A$ and
- ϕ angle of rotation of link 2 about $\hat{u}B$, both measured from their respective unknown starting positions, to be determined in the synthesis.

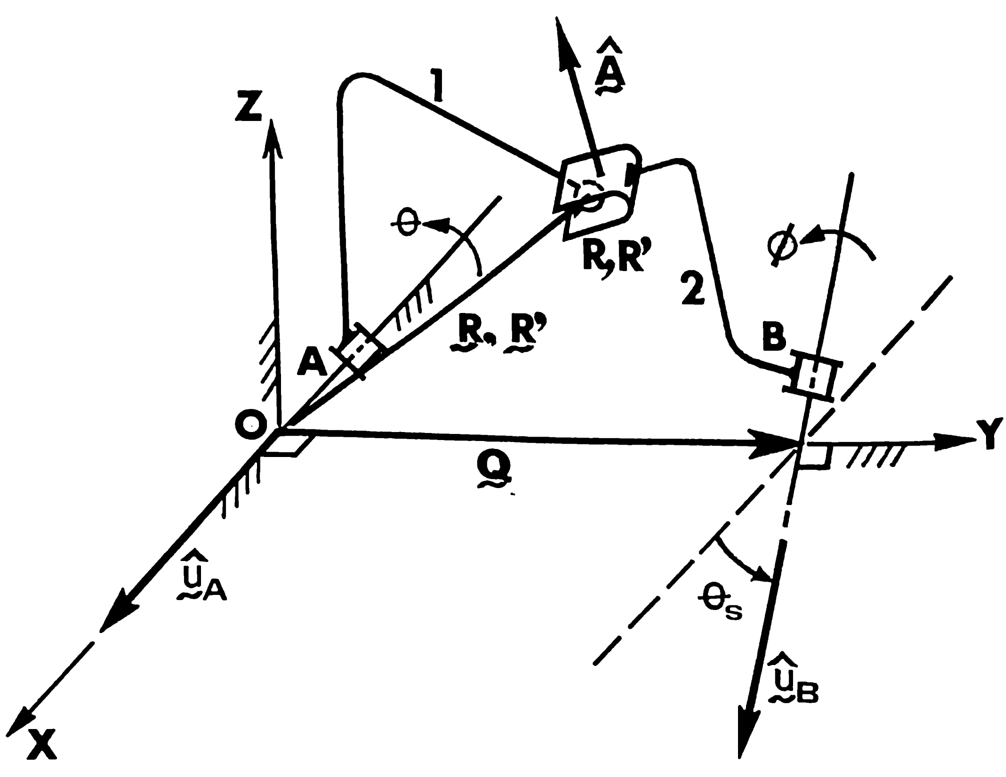

Figure 1. Schematic of the R-Sp-R 3-link spatial function generator mechanism.

The vectors $\underset{\sim}{R}$ and $\underset{\sim}{\hat{A}}$ are the design vectors of the mechanism. The other vectors simply position and orient the mechanism in the coordinate system.

By prescribing up to five motion specifications for the mechanism, a system of non-linear equations is obtained consisting of up to six equations. If j = 1 denotes the initial position of the mechanism, j = 2 the second, and so on, then the synthesis equations obtained are in the following form (the reader is referred to references [14] and [16] for the details of the derivation of the synthesis equations):

$$[(1 + pd_2)k_{1j} + k_{2j}d_1]r_1 + [K_{3j} + k_{4j}d_1 + k_{5j}d_2]r_2$$
$$+ [k_{6j} + k_{7j}d_1 + k_{8j}d_2]r_3 + [(1 + pd_2)k_{2j} + qk_{1j}d_1] = 0$$
$$j = 2, 3, \text{up to } 6 (j - 1 \text{ equations}) \tag{1}$$

where

p, q, and the K_{ij}'s are constants, determined from the motion specifications, are known and r_1, r_2, r_3 and d_1, d_2 are the unknown parameters of the design vectors, namely vector $\underset{\sim}{R}$ and unit vector $\underset{\sim}{\hat{A}}$ respectively.

The Elimination Method

The system of equations (1) is solved for r_1, r_2, r_3 and d_1, d_2 by the elimination method. This method is a procedure of obtaining from the system a polynomial in only one variable, the so-called "eliminant variable" (the rest of the variables are eliminated). The roots of this "eliminate" equation are the solutions for that one variable and are then used to obtain the solutions to the other variables by back substitution.

The procedure for getting the eliminant polynomial consists of setting up a homogeneous system of n linear equations in n unknowns from the original system of (j – 1) non-linear equations (1). The coefficients of the unknowns are made to be in terms of one variable, the variable in the eventual eliminant polynomial. The unknowns may include constants, functions, products and/or powers of the other variables to be eliminated. Depending on the number of prescribed positions and the combinations of unknown variables, the system of homogeneous equations may be obtained either directly from the system of equations or by multiplying the system of equations with appropriately chosen variable or variables and adding these new equations to the original system. This may be done a number of times until the number of unknowns and number of equations are equal. The eliminant polynomial is then obtained by setting the determinant of the coefficient matrix of the homogeneous system of equations to zero. The resulting square matrix of the coefficients must be singular to assure that the system of equations will have a set of non-zero simultaneous solutions for the unknowns. In other words, it insures that the system of equations is compatible, therefore, the principal determinant of the matrix of coefficients is equal to zero and the roots of the resulting polynomial are compatible values of the eliminant variable. The method will now be illustrated by the following two examples:

Example 1: Three equations in three unknowns with two product terms.

$$a_j x + b_j y + c_j z + d_j xy + e_j xz + f_j = 0$$
$$j = 1, 2, 3$$

The equations are re-arranged so that the coefficients are in terms of one variable, say x. Thus,

$$(b_j + d_j x)y + (c_j + e_j x)z + (f_j + a_j x) = 0$$
$$j = 1, 2, 3 \tag{3}$$

Equations (3) are now considered as a homogeneous system of three equations in the three variables y, z and 1. Thus, the determinant of the coefficient matrix must be zero to have non-trivial solutions. This yields the eliminant as the determinant

$$\left|(b_j + d_jx)\ (c_j + e_jx)\ (f_j + a_jx)\right| = 0 \tag{4}$$

Equation (4) is called the eliminant polynomial – whereby y, z and 1 were eliminated from the system of equations and x is forced to take on values such that it will be common to all the three equations of equation (2) or (3). Equation (4) expands into a cubic equation in x, the roots of which are solutions for x. To find the solutions for y and z, these roots of x are then substituted into any two equations of (2) or (3), giving a system of two linear, non-homogeneous equations.

Example 2: Three equations in three unknowns with three product terms.

$$a_jx + b_jy + c_jz + d_jxy + e_jxz + f_jyz + g_j = 0$$
$$j = 1, 2, 3 \tag{5}$$

One variable (again x) is chosen as the eliminant variable and equation (5) is rearranged as

$$(g_j + a_jx)1 + (c_j + e_jx)z + (b_j + d_jx)y + f_jyz = 0$$
$$j = 1, 2, 3 \tag{6}$$

The three equations (6) can be considered as having four unknowns: variables 1, z, y and yz. Multiplying these three equations by y will give three more equations and also introduce two additional unknowns, y^2 and y^2z. The system of six equations in the six unknowns 1, y, z, yz, y^2 and y^2z is written as

$$\begin{pmatrix} (g_j + a_jx) & (c_j + e_jx) & (b_j + d_jx) & f_j & 0 & 0 \\ 0 & 0 & (g_j + a_jx) & (c_j + e_jx) & (b_j + d_jx) & f_j \end{pmatrix} \begin{pmatrix} 1 \\ y \\ \vdots \\ \vdots \\ y^2z \end{pmatrix} = \underset{\sim}{0}$$
$$j = 1, 2, 3 \tag{7}$$

The method shown in obtaining the coefficient matrix is called Sylvester's Dyalitic Method of Elimination [17]. Other methods exist but Sylvester's method is straightforward and does not require manipulations of the equations. The determinant of the coefficient matrix is now set to zero, expanding to the eliminant polynomial in x. Its roots are then substituted into any five equations of (7) to get the values of z, y and even yz, y^2 and y^2z. Consistency of these roots with one another is an excellent check on computations.

The method will now be applied to solve two synthesis cases for the R-Sp-R mechanism – the five-position problem where the unknowns are r_1, r_2, r_3 and d_1, and the six-position problem where the unknowns are r_1, r_2, r_3, d_1 and d_2.

Five-Position Synthesis of the R-Sp-R

With d_2 [eq. (1)] known, the eliminant can readily be determined from system (1) written for j = 2, 3, 4 and 5. This eliminant is written as the following determinant set to zero:

$$\left|[(1 + pd_2)k_{1j} + k_{2j}d_1](A_j + B_jd_1)(C_j + D_jd_1)[(1 + pd_2)k_{2j} + qk_{1j}d_1]\right| = 0$$
$$j = 2, 3, 4, 5 \tag{8}$$

The eliminant (8) would seem to be a quartic polynomial initially. However, because of the dependency of the terms in the first and fourth columns of the determinant, a constant polynomial independent of motion specifications can be factored out. This is the polynomial,

$$(1 + pd_2)^2 - qd_1 2 \tag{9}$$

Thus, the quadratic eliminant polynomial is obtained as

$$|k_{1j}\,(A_j + B_j d_1)\,(C_j + D_j d_1)\,k_{2j}| = 0 \tag{10}$$

The 2 or 0 real roots (only real roots are acceptable as solutions) for d_1 are then substituted into any three equations of system (1). This will result in a system of three linear, non-homogeneous equations in unknowns r_1, r_2 and r_3.

Six-Position Synthesis of the R-Sp-R

Because of the relationship among some of the coefficients in the synthesis equation (1), a dyalitic elimination applied directly on the design variables will result in a null polynomial for the eliminant Furthermore, the system of equations has the polynomial of equation (10) as a common factor, so that application of dyalitic elimination is redundant – the condition that common roots exist is already satisfied. To obtain an eliminant, the synthesis equations are arranged so that all the coefficients are independent and in terms of one variable d_2. Thus

$$(k_{3j} + k_{5j} d_2)\, r_2 + (k_{6j} + k_{8j} d_2)\, r_3 + k_{1j}\, x + k_{2j} y + k_{4j} d_1 r_2 + k_{7j} d_1 r_3 = 0$$
$$j = 2, 3, 4, 5, 6 \tag{11}$$

where the auxiliary variables x and y are

$$x = (1 + pd_2)\, r_1 + qd_1$$
$$y = (1 + pd_2) + d_1\, r_1 \tag{12}$$

Equation (12) is now considered to have six unknowns r_2, r_3, x, y, $d_1 r_2$ and $d_1 r_3$ in five equations. Multiplying the five equations by d_1 will introduce four additional unknowns $d_1 x$, $d_1 y$, $d_1{}^2 r_2$ and $d_1{}^2 r_3$. Thus the eliminant is obtained from the following 10x10 determinant set to zero.

$$\begin{vmatrix} (K_{3j} + K_{5j} d_2) & (K_{6j} + K_{8j} d_2) & K_{1j} & K_{2j} & K_{4j} & K_{7j} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (K_{3j} + K_{5j} d_2) & (K_{6j} + K_{8j} d_2) & k_{1j} & k_{2j} & k_{4j} & k_{7j} \end{vmatrix} = 0$$
$$j = 2, 3, 4, 5, 6 \tag{13}$$

The eliminant turns out to be a cubic and will give one or three real roots for d_2.

The next step is to obtain the values of d_1. One can use the values of d_2 and substitute them into any four of the five equations and then get an eliminant in d_1 (see the five-position problem). This method will, however, introduce and extraneous root. One of the roots that will be obtained from the quadratic in d_1 will satisfy only the four equations used in the eliminant for d_1. The other root for d_1 will be the value of d_1 that will satisfy all the five equations. This method could be used, but a better method would be to set up an eliminant in d_1 which will give only the one correct d_1. This is achieved by using all the five equations at the same time.

The synthesis equations (d_2 already known) can be arranged in the following form.

$$K_{1j}x + [(K_{3j} + k_{5j}d_2) + K_{4j}d_1]r_2 + (K_{6j} + K_{8j}d_2)r_3 + K_{7j}d_1r_3 + k_{2j}y = 0$$

$$j = 2, 3, 4, 5, 6 \qquad (14)$$

From equations (14) the eliminant is obtained as the determinant

$$|K_{1j}\ [(K_{3j} + k_{5j}d_2) + k_{4j}d_1]\ (K_{6j} + k_{8j}d_2)\ K_{7j}\ k_{2j}| = 0$$

$$j = 2, 3, 4, 5, 6 \qquad (15)$$

The value of d_1 obtained from (15) and the corresponding d_2 used are then substituted back into any three equations of (2) to get r_1, r_2 and r_3.

From the preceding application of the elimination method, one can make the following general observations.

1. Having some dependency on the coefficients of the equations will make a dyalitic elimination impractical.
2. One can eliminate an unknown variable or its function (considered a different unknown) even when the same variable is used in the eliminant (see eq. 12).
3. Anytime a system of equations is multiplied by a function of one of the unknowns, extraneous roots may be introduced.

The RSSR-SC Spatial Motion Generator

Figure 2 shows a schematic of the five-link RSSR-SC spatial motion generator. It is required that its coupler, the SSS link, be guided through four finitely or infinitesimally separated arbitrarily prescribed positions. The formulation of the synthesis equations results in the following two sets of non-linear systems of equations (see reference [15] for the formulation of the synthesis equations).

Set A

$$\sum_{i=0}^{2} x_i(L_{i1j} + L_{i2j}y_1 + L_{i3j}y_2) = 0$$

$$x_0 = 1 \qquad j = 2, 3, 4, \qquad (16)$$

and

$$\sum_{j=0}^{3}\left(\sum_{i=0}^{3-j} a_{ij}x_1^{\,i}\right)x_2^{\,j} = 0 \qquad (17)$$

Set B

$$\sum_{i=0}^{1} u_i(M_{i1j} + M_{i2j}u_1 + M_{i3j}v_1 + M_{i4j}v_2) = 0$$

$$u_0 = 1 \qquad j = 2, 3, 4, \qquad (18)$$

where: The variables x_1, x_2; y_1, y_2 and v_1, v_2 are any two coordinates of the unknown vectors $\underset{\sim}{\alpha}$, $\underset{\sim}{a}_0$ and $\underset{\sim}{b}_0$ respectively, and the variable u_1 is any one coordinate of the unknown vector $\underset{\sim}{\beta}$. The coefficients, L's, a's and M's, which are deeerministic functions of the prescribed motion parameters, are thus known.

Sandor, Kohli, Zhuang and Reinholtz in reference [15] reduced the foregoing equations (Set A) to two simultaneous cubic equations which were solved numerically. Set B yielded a 4-th degree polynomial, which was solved in closed form.

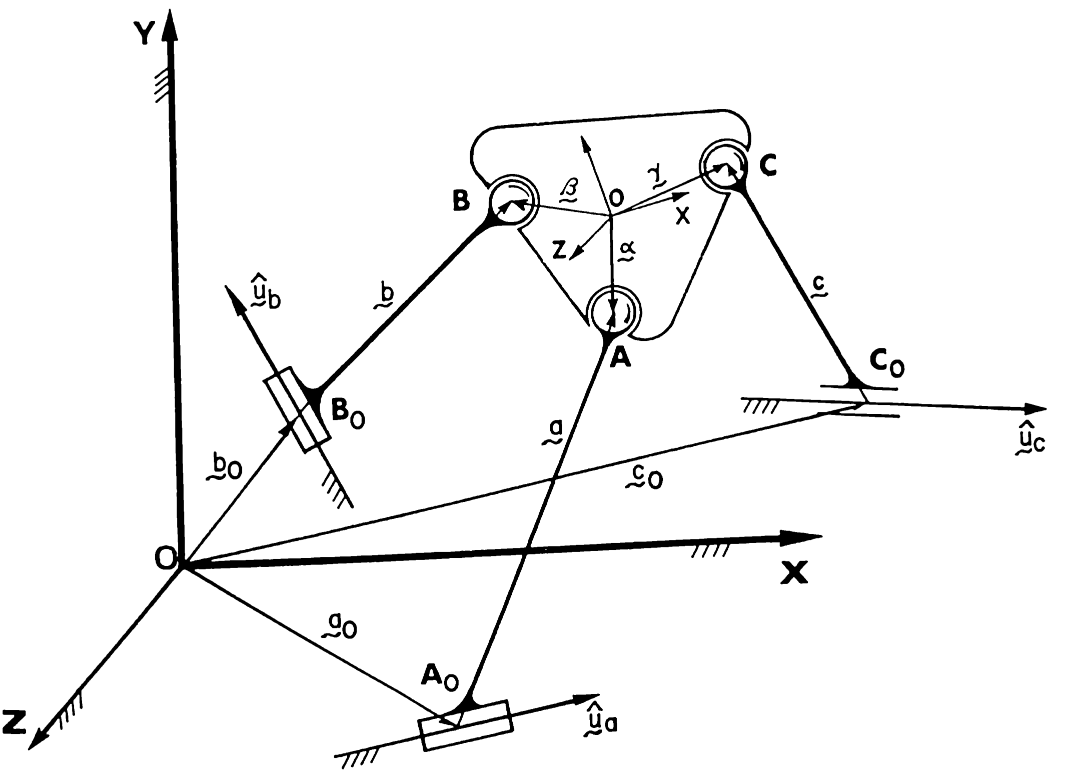

Figure 2. Schematic of the RSSR-SC 5-link spatial motion generator mechanism.

The systems of equations given by sets A and B can also be solved by the elimination method. For systems A, the unknown variables y_1 and y_2 are first eliminated from equation (16) by obtaining an eliminant in x_1 and x_2. This eliminant is cubic in x_1 and x_2 and is in the same form as equation (17). The variable x_1 or x_2 is then eliminated from these two cubic equations to give a ninth degree polynomial in x_1 or x_2.

The solution to equation (18) is similar to that shown in Example 2.

Dyalitic Elimination of 2 Cubic Equations

From set A (equations (17) and (18), the two cubic equations in x_1 and x_2 can be written as

$$A_1 x_1^3 + B_1 x_1^2 + C_1 x_1 + D_1 = 0$$

and

$$A_2 x_1^3 + B_2 x_1 + C_2 x_1 + D_2 = 0 \qquad (19)$$

Where
$$A_j = a_j$$
$$B_j = b_{1j} x_2 + b_{2j}$$
$$C_j = c_{1j} x_2^2 + c_{2j} x_2 + c_{3j}$$
$$D_j = d_{1j} x_2^3 + d_{2j} x_2^2 + d_{3j} x_2 + d_{4j}$$
$$j = 1, 2 \qquad (20)$$

Multiplying both equations of (19) by x_1 and x_1^2 will give the following eliminant:

$$\begin{vmatrix} 0 & 0 & A_1 & B_1 & C_1 & D_1 \\ 0 & A_1 & B_1 & C_1 & D_1 & 0 \\ A_1 & B_1 & C_1 & D_1 & 0 & 0 \\ 0 & 0 & A_2 & B_2 & C_2 & D_2 \\ 0 & A_2 & B_2 & C_2 & D_2 & 0 \\ A_2 & B_2 & C_2 & D_2 & 0 & 0 \end{vmatrix} = 0 \qquad (21)$$

This will give the ninth degree eliminant polynomial in x_2 -- the roots of which are the solutions for x_2. The solutions for x_1 are then obtained by finding the identical roots of the two equations of (19).

Conclusions

This paper demonstrates the applicability of the elimination method in solving non-linear equations involved in the synthesis of some spatial mechanisms. This method has the advantage of revealing the maximum number of solutions possible. A disadvantage of this method, however, is that in almost all cases, one has to adapt the method to the specific problem. The procedure is sensitive to manipulation and arrangement of the terms in the equations of the system.

Acknowledgment

The authors wish to acknowledge the support of this research under NSF Grant No. MEA-8025812 sponsored at the University of Florida by the Mechanical System Program, Civil and Mechanical Engineering Division of the National Science Foundation. The authors are grateful to Susan Conway for her expert typing.

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