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An Integral-Momentum Analysis of the Laminar Boundary Layer along a Rectangular Corner

by

Leonardo Q. Liongson, Ph.D.*

Abstract

An integral-momentum analysis is applied to the problem of the laminar boundary layer along a rectangular corner. It involves an assumed self-similar velocity profile defined by functions with prescribed properties. Two case examples are provided. Reasonably good agreement is obtained between the results of the present study and the available numerical solution of previous workers on the problem.

Introduction

The viscous flow along a rectangular corner is characterized by the presence of three regions depicted in Figure 1: a corner boundary layer, a pair of plane boundary layers, and a potential flow or free stream region. The two plane layers mutually interact in the corner layer so that the latter also grows in area as the plane layers thicken in the downstream x-direction. A three-dimensional secondary or cross flow is also induced.

Numerical solutions of the partial differential equations governing laminar corner layers were obtained by Carrier (1947) and later, with an improved scheme, by Rubin and Grossman (1971). Rubin (1966) derived a solution by the method of matched asymptotic expansion. A numerical solution for the turbulent corner layer was developed by Shafir and Rubin (1976).

The present study aims to provide an approximate solution to the laminar corner layer problem by means of a three-dimensional integral-momentum analysis which involves an assumed self-similar velocity profile for the x-component of velocity. Specifically, the integral formulation seeks to predict the width w(x) of the corner layer in relation to the thickness $\delta(x)$ of the plane layers which has been predicted by the two-dimensional integral momentum approximation found in the literature (Schlichting, 1979).

Self-Similar Velocity Profile

With reference to Figure 2, the assumed self-similar profile for the x-component of velocity, u(x, y, z), is prescribed as follows:

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*Chairman, Department of Engineering Sciences, U.P. College of Engineering.

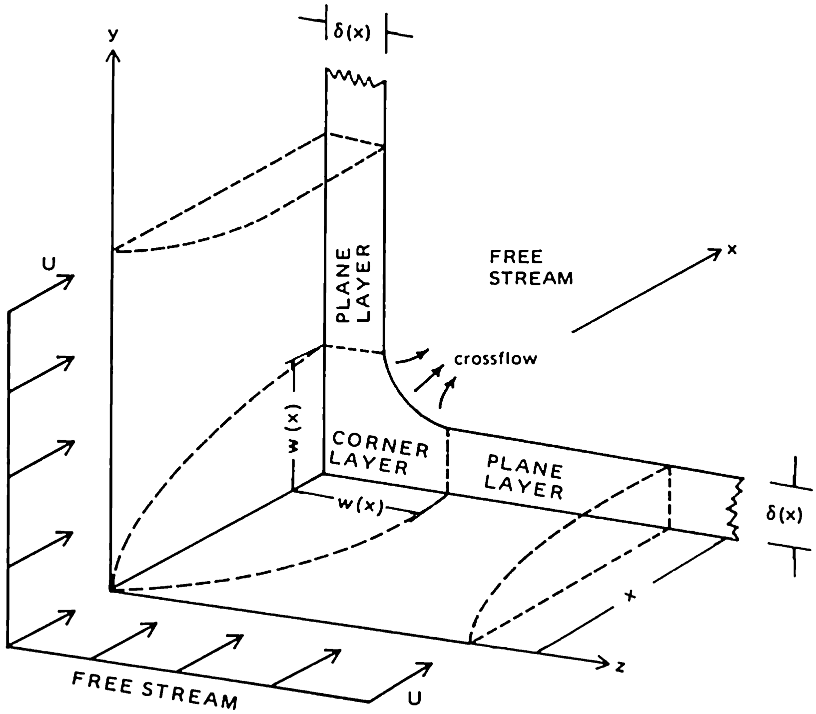

Figure 1. The three regions of flow along a rectangular corner

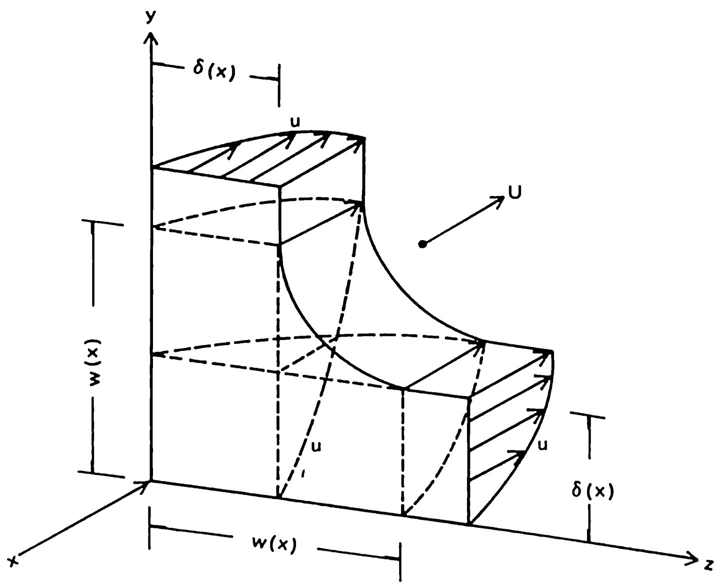

Figure 2. The x-component velocity profile, u(x,y,z)

(a) In the plane layer attached to the xz-plane:

$$u = U\, f[y/\delta(x)] \tag{1}$$

$$0 \leqslant y \leqslant \delta(x); \; z \geqslant w(x).$$

(b) In the plane layer attached to the xy-plane:

$$u = U\, f[z/\delta(x)] \tag{2}$$

$$0 \leqslant z \leqslant \delta(x); \; y \geqslant w(x).$$

(c) In the corner layer, A (x):

$$u = U\, f\left[\frac{q\{p(y/w(x))\; p(z/w(x))\}}{q\{p(\delta(x)/w(x))\}}\right]$$

$$= U\, f\left(\frac{1}{r}\, q\{\, p(y/w(x))\; p(z/w(x))\}\right) \tag{3}$$

wherein the domain A(x) of (y,z) is defined by
$p(y/w(x))\; p(z/w(x)) \leqslant p\,(\delta\,(x)/w(x)\,)$
and $0 \leqslant y,\, z \leqslant w(x)$
and r is the ratio $\delta(x)/w(x)$ assumed to be constant.

(d) In the free-stream region, outside of both corner and plane layers:

$$u = U, \text{ a uniform free-stream velocity.} \tag{4}$$

The required properties of the dimensionless functions f, p, and q are given below:

(a) Properties of f

1. $f(0) = 0$ (No slip condition on a solid boundary).
2. $f(1) = 1$ and $f'(1) = 0$ (First-order continuity of the boundary layer flow with the free stream).
3. $f'(0) > 0$ (Positive shear on a solid boundary).
4. $f(\alpha)$ is continuous and monotonically increasing in $0 < \alpha < 1$.
5. $f''(\alpha) < 0$ in $0 < \alpha < 1$ (Absence of inflection point).

(b) Properties of p and q

1. $p(0) = 0$ (No slip condition on a solid boundary).
2. $p(1) = 1$ (Zeroth-order continuity of the corner layer flow with the plane layer flow).
3. $p(\alpha)$ is continuous and monotonically increasing in $0 < \alpha < 1$.
4. q is the inverse of p.
5. $p'(0) = 1/q'(0) > 0$ (Positive shear on a solid boundary).

According to Equation 3, the assumed isovelocity contours or isovels inside the corner layer are described by the product of functions $p(y/w(x))\; p(z/w(x))$ = constant, which is symmetric in y and z. From the properties of p and q, it can be easily verified that Equation 3 matches with both Equations 1 and 2 along the common boundaries of the corner layer with the two plane layers.

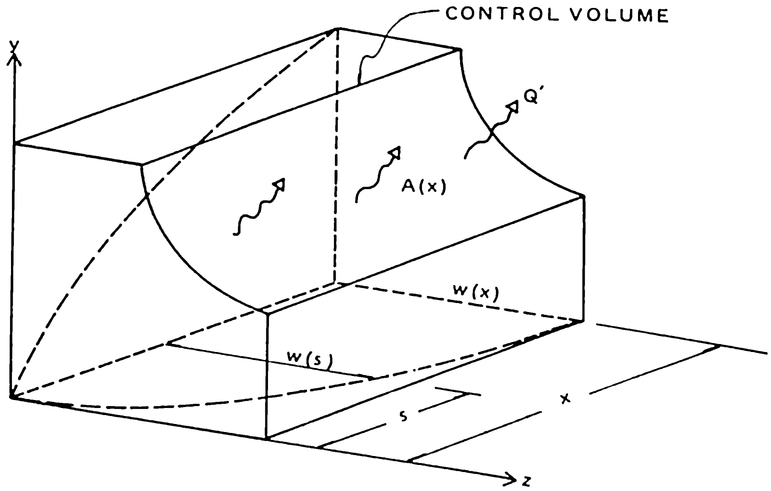

Figure 3. The control volume for the integral formulation

Integral Formulation

A control volume defined by $0 \leqslant s \leqslant x$ and $(y,z) \epsilon$, $A(x)$ and sketched in Figure 3 is adopted for the integral formulation of the mass and momentum conservation laws.

Defining Q' as the volumetric rate of cross flow which leaves the control volume, the incompressible-flow continuity equation is written in integral form as

$$U \iint_{A(x)} dy\, dz = Q' + \iint_{A(x)} u(x,y,z)\, dy\, dz$$

(free-stream inflow) = (cross flow) + (corner layer outflow)

Introducing the dimensionless coordinates $\alpha = y/w(x)$ and $\beta = z/w(x)$ and the self-similar velocity profile, the continuity equation becomes

$$U w^2(x) \iint_A d\alpha\, d\beta = Q' + U w^2(x) \iint_A f[\frac{1}{r} q\{p(\alpha)\, p(\beta)\}]\, d\alpha\, d\beta$$

Hence,

$$Q' = U w^2(x)\ [E(r) - F(r)] \tag{5}$$

where $E(r) = \iint_A d\alpha\, d\beta$

$$F(r) = \iint_A f[\frac{1}{r} q\{p(\alpha)\, p(\beta)\}]\, d\alpha\, d\beta$$

A = domain of (α,β) in the corner layer.

The x-component integral-momentum equation applicable to the control volume is

$$\iint_{A(x)} u^2 (x,y,z)\, dy\, dz + Q'U - U^2 \iint_{A(x)} dy\, dz$$

$$= - \int_0^x \left\{ \int_0^{w(x)} \nu \frac{\partial u}{\partial z} \bigg|_{z=0} dy + \int_0^{w(x)} \nu \frac{\partial u}{\partial y} \bigg|_{y=0} dz \right\} ds \qquad (6)$$

The terms on the left-hand side of Equation 6 are the momentum efflux through the corner layer, the momentum efflux in the cross flow, and the momentum influx of the free stream, respectively. The right-hand side of Equation 6 represents the total viscous shear force exerted on the fluid by the two perpendicular flat plates. The pressure gradient is taken to be zero.

Upon substitution of the dimensionless coordinates, the self-similar profile, and the value of Q' (Equation 5) in Equation 6, the latter becomes

$$U^2 w^2 (x) \iint_A f^2 \left[\frac{1}{r} q \{p(\alpha) p(\beta)\} \right] d\alpha\, d\beta + U^2 w^2(x) \{E(r) - F(r)\}$$

$$- U^2 w^2 (x) \iint_A d\alpha\, d\beta = - \int_0^x \left[\int_0^{w(s)} \nu U f'(0)\, q'(0)\, p'(0) \frac{1}{r} p \left(\frac{y}{w(s)} \right) \frac{dy}{w(s)} \right.$$

$$+ \int_{w(s)}^{w(x)} \nu U f'(0) \frac{1}{r} \frac{dy}{w(s)} + \int_0^{w(s)} \nu U f'(0) q'(0)\, p'(0) \frac{1}{r} p \left(\frac{z}{w(s)} \right) \frac{dz}{w(s)}$$

$$\left. + \int_{w(s)}^{w(x)} \nu U f'(0) \frac{1}{r} \frac{dz}{w(s)} \right] ds$$

The above equation further simplifies to

$$U^2 w^2(x) \{ F(r) - G(r) \} = 2 \nu U H \frac{1}{r} \int_0^x \{I_p - 1 + \frac{w(x)}{w(s)}\} \quad ds \qquad (7)$$

where $G(r) = \iint_A f^2 \left[\frac{1}{r} q \{p(\alpha)\, p(\beta)\} \right] d\alpha\ d\beta$

$$H = f'(0)\, q'(0)\ p'(0) = f'(0)$$

$$I_p = \int_0^1 p(\alpha)\, d\alpha.$$

The result given by Equation 7 has to be matched with the result obtained by the two-dimensional integral momentum analysis of the plane boundary layer (Schlichting, 1979, p. 204):

$$\delta(x) = \sqrt{\frac{2 \nu H x}{U I_1}} \qquad (8)$$

where $I_1 = \int_0^1 f(\gamma) [1 - f(\gamma)]\, d\gamma$.

Replacing $\delta(x)$ by $rw(x)$ in Equation 8 gives

$$w(x) = \sqrt{\frac{2\nu Hx}{UI_1 r^2}} \tag{9}$$

It follows from Equation 9 that

$$\frac{w(x)}{w(s)} = \sqrt{\frac{x}{s}}$$

which may be substituted on the right-hand side of Equation 7.

Integrating the right-hand side of Equation 7 yields

$$w^2(x)\ \{F(r) - G(r)\} = \frac{2\nu H}{Ur}\ (I_p + 1)\, x$$

which gives

$$w(x) = \sqrt{\frac{2\nu H\,(I_p + 1)\, x}{U\,[F(r) - G(r)]\, r}} \tag{10}$$

Equating the right-hand sides of Equations 9 and 10 provides the following equation in the unknown $r = \delta(x)/w(x)$:

$$F(r) - G(r) = (I_p + 1)\, I_1\, r. \tag{11}$$

Equation 11 is solved for r in the next sections for assumed forms of the functions p, q, and f.

Case 1. $p(\alpha) = \alpha$, $q(\alpha) = \alpha$, arbitrary f.

The isovels inside the corner layer under this case are hyperbolas given by yz = constant. The following integrals are evaluated:

$$I_p = \int_0^1 p(\alpha)\, d\alpha = \int_0^1 \alpha\, d\alpha = \frac{1}{2}$$

$$\begin{aligned} F(r) - G(r) &= \int_{\beta=0}^{\beta=r} \int_{\alpha=0}^{\alpha=1} f(\alpha\beta/r) \left[1 - f(\alpha\beta/r)\right] d\alpha\, d\beta \\ &+ \int_{\beta=r}^{\beta=1} \int_{\alpha=0}^{\alpha=r/\beta} f(\alpha\beta/r) \left[1 - f(\alpha\beta/r)\right] d\alpha\, d\beta \\ &= r\left[I_2 + I_1\ \ln(1/r)\right] \end{aligned}$$

where $I_2 = \int_0^1 \int_0^1 f(\alpha\beta) \left[1 - f(\alpha\beta)\right] d\alpha\, d\beta.$

Applying the above integrals to Equation 11 yields the expression for r:

$$r = \exp\,[I_2/I_1 - 3/2]. \tag{12}$$

The area of the corner layer is easily evaluated as

$$A_c = \delta^2 (x) \frac{1}{r} \left\{1 + \ln (1/r)\right\} \tag{13}$$

Case 2. $p(\alpha) = \sin (\frac{1}{2} \pi\alpha)$, $q(\alpha) = \frac{2}{\pi}$ arcsin (α), arbitrary f.

The isovels inside the corner layer are quarter ovals given by $\sin (\frac{1}{2} \pi\, y/w(x))$ $\sin (\frac{1}{2} \pi\, z/w(x)) =$ constant. Since $p(1) = 1$ and also $p'(1) = 0$, the corner layer flow is continuous up to the first order with the plane layer flow. The following integrals are evaluated:

$$I_p = \int_o^1 p(\alpha)\, d\alpha = \int_o^1 \sin (\tfrac{1}{2} \pi\alpha)\, d\alpha = 2/\pi$$

The integral $[F(r) - G(r)]$ is more conveniently evaluated by first transforming the orthogonal coordinates from (α,β) to (λ, η):

$$\sin (\lambda) = \sin (\tfrac{1}{2} \pi\, \alpha) \sin (\tfrac{1}{2} \pi\, \beta) \quad (0 < \lambda < \tfrac{1}{2} \pi\, r)$$

$$\eta = \cos (\tfrac{1}{2} \pi\, \alpha)/\cos (\tfrac{1}{2} \pi\, \beta) \quad (0 < \eta < \infty)$$

The Jacobian determinant $J = \partial(\lambda,\eta)/\partial(\alpha,\beta)$ can be derived as

$$J = \frac{\pi^2}{4} \sqrt{(\eta^2 + 1)^2 - 4\, \eta^2 \cos^2 \lambda} \,/ \cos \lambda\, .$$

Hence,

$$F(r) - G(r) = \int_{\lambda=o}^{\lambda=\frac{1}{2}\pi r} f\left(\frac{2\lambda}{\pi r}\right) \left[1 - f\left(\frac{2\lambda}{\pi r}\right)\right] \int_{\eta=o}^{\eta=\infty} \frac{d\eta\, d\lambda}{J}$$

But $\int_{\eta=o}^{\eta=\infty} \frac{d\eta}{J}$ can be shown to be expressible

as $\int_{\eta=o}^{\eta=\infty} \frac{d\eta}{J} = \frac{4}{\pi^2} \cos \lambda\ K(\cos^2 \lambda)$

where $K(\cos^2\lambda)$ = complete elliptic integral of the first kind with parameter $\cos^2 \lambda$:

$$K(\cos^2 \lambda) = \int_o^{\pi/2} \frac{d\theta}{\sqrt{1 - \cos^2\lambda \sin^2\theta}}$$

(Milne-Thomson, 1950).

Substitution of the above integrals in Equation 11 gives the following implicit equation for the unknown r (with λ replaced by $\frac{1}{2} \pi\, \gamma\, r$):

$$\int_{\gamma=o}^{\gamma=1} f(\gamma) \left[1 - f(\gamma)\right] \cos\left(\frac{1}{2}\pi\gamma r\right) K \left[\cos^2\left(\frac{1}{2}\pi\gamma r\right)\right] \left(\frac{1}{2}\pi\, d\gamma\right)$$

$$= \frac{\pi^2}{4}\left(\frac{2}{\pi} + 1\right) I_1 . \tag{14}$$

The integral on the left-hand side of Equation 14 was numerically evaluated for trial values of r by means of the trapezoidal rule, with discrete interval $\Delta\gamma = 0.02$, utilizing linearly interpolated values of K from Milne-Thomson's (1950) table. For an assigned function f, the correct value of r was located by interactive searching with a microcomputer.

The area of the corner layer is evaluated from

$$A_c = \delta^2 (x) \frac{4}{r\pi^2} \int_{\gamma=o}^{\gamma=1} \cos\left(\frac{1}{2}\pi\gamma r\right) K \left[\cos^2\left(\frac{1}{2}\pi\gamma r\right)\right] \left(\frac{1}{2}\pi d\gamma\right) \tag{15}$$

using the same numerical quadrature.

Table 1

Results of Computations

f (α)	Case 1 p (α) = α Ip = 1/2			Case 2 p (α) = sin ($\frac{1}{2}\pi\ \alpha$) Ip = $2/\pi$		
	r	R	$Ac/\delta^2(x)$	r	R	$Ac/\delta^2(x)$
sin ($\frac{1}{2}\pi\ \alpha$)	0.6651	1.5035	2.1166	0.5396	1.8534	2.8486
$\frac{3}{2}\alpha - \frac{1}{2}\alpha^3$	0.6489	1.5411	2.2076	0.5265	1.8993	2.9553
$2\alpha - \alpha^2$	0.7110	1.4065	1.8860	0.5757	1.7370	2.5823
$2\alpha - 2\alpha^3 + \alpha^4$	0.7863	1.2718	1.5775	0.6362	1.5719	2.2095

f (α)	I_1	I_2	$\delta(x)/\sqrt{\nu x/U}$
sin ($\frac{1}{2}\pi\ \alpha$)	0.1366	0.1492	4.7957
$\frac{3}{2}\alpha - \frac{1}{2}\alpha^3$	0.1393	0.1487	4.6407
$2\alpha - \alpha^2$	0.1333	0.1545	5.4779
$2\alpha - 2\alpha^3 + \alpha^4$	0.1175	0.1480	5.8346

Table 2
Values of u/U

I. From Rubin and Grossman (1966):

$y/\sqrt{2\nu x/U}$ \ $z/\sqrt{2\nu x/U}$	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2	8.0	8.8
8.8											1.0
8.0										1.0	1.0
7.2									1.0	1.0	1.0
6.4								1.0	1.0	1.0	1.0
5.6							1.0	1.0	1.0	1.0	1.0
4.8						.998	.999	1.0	1.0	1.0	1.0
4.0					.966	.989	.994	.996	.996	.997	.997
3.2				.815	.905	.950	.966	.971	.974	.975	.976
2.4			.543	.677	.781	.842.	.869	.880	.885	.888	.890
1.6		.263	.381	.487	.573	.629	.657	.670	.677	.680	.682
0.8	.068	.134	.195	.251	.297	.329	.346	.355	.359	.361	.362
0.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0

II. $p(\alpha) = \alpha \,;\, f(\alpha) = \frac{3}{2}\alpha - \frac{1}{2}\alpha^3$

$y/\sqrt{2\nu x/U}$ \ $z/\sqrt{2\nu x/U}$	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2	8.0	8.8
8.8											1.0
8.0										1.0	1.0
7.2									1.0	1.0	1.0
6.4								1.0	1.0	1.0	1.0
5.6							1.0	1.0	1.0	1.0	1.0
4.8						1.0	1.0	1.0	1.0	1.0	1.0
4.0					.998	1.0	1.0	1.0	1.0	1.0	1.0
3.2				.808	.928	.992	.999	.999	.999	.999	.999
2.4			.500	.645	.771	.874	.901	.901	.901	.901	.901
1.6		.230	.341	.448	.550	.645	.673	.673	.673	.673	.673
0.8	.058	.115	.173	.230	.286	.341	.358	.358	.358	.358	.358
0.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0

III. $p(\alpha) = \sin(\frac{1}{2}\pi\alpha)$; $f(\alpha) = \frac{3}{2}\alpha - \frac{1}{2}\alpha^3$

$y/\sqrt{2\nu x/U}$											
8.8											1.0
8.0										1.0	1.0
7.2									1.0	1.0	1.0
6.4								1.0	1.0	1.0	1.0
5.6							1.0	1.0	1.0	1.0	1.0
4.8						1.0	1.0	1.0	1.0	1.0	1.0
4.0					.998	1.0	1.0	1.0	1.0	1.0	1.0
3.2				.849	.941	.984	.998	.999	.999	.999	.999
2.4			.566	.700	.798	.861	.894	.901	.901	.901	.901
1.6		.277	.398	.500	.579	.635	.666	.673	.673	.673	.673
0.8	.073	.142	.206	.260	.304	.336	.354	.358	.358	.358	.358
0.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0
	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2	8.0	8.8

$z/\sqrt{2\nu x/U}$

Discussion of Results

The computed values of r, R=1/r, and $A_C/\ \delta^2(x)$ under Cases 1 and 2 are given in Table 1, for the various choices of f found in the literature (Schlichting, 1979):

(a) $f(\alpha) = \sin(\frac{1}{2}\pi\alpha)$

(b) $f(\alpha) = \frac{3}{2}\alpha - \frac{1}{2}\alpha^3$

(c) $f(\alpha) = 2\alpha - \alpha^2$

(d) $f(\alpha) = 2\alpha - 2\alpha^3 + \alpha^4$

Table 1 also gives the values of I_1, I_2, and $\delta(x)/\sqrt{\nu x/U}$ for the different choices of f.

It is observed that the value of r is very sensitive to the choice of the function p and, to a lesser extent, to the choice of function f. Under Case 1, $p(\alpha) = \alpha$, r ranges from 0.6489 to 0.7863, depending on the choice of f. Under Case 2, $p(\alpha) = \sin(\frac{1}{2}\pi\alpha)$, r ranges from 0.5265 to 0.6362.

Reasonably good agreement is obtained between the present results for $f(\alpha) = \frac{3}{2}\alpha - \frac{1}{2}\alpha^3$ and the numerical solution of Rubin and Grossman (1966). Table 2 reproduces their values of u/U for $0 \leqslant y/\sqrt{2\nu x/U} \leqslant z/\sqrt{2\nu x/U} \leqslant 8.8$, and also provides the values of u/U taken from the present study for $f(\alpha) = \frac{3}{2}\alpha - \frac{1}{2}\alpha^3$ under both Cases 1 and 2.

Conclusion

The three-dimensional integral-momentum formulation has been shown to be effective in providing approximate solution to the corner layer problem. The higher sensitivity of the ratio $\delta(x)/w(x)$ to the assumed function p was observed. The observed moderate sensitivity to the function f is consistent with findings in two-dimensional analysis. Good agreement was obtained between the u/U values computed in this study and those of Rubin and Grossman (1966). The positive results should encourage the application of the integral-momentum formulation to other three-dimensional boundary layer problems.

Notation

A	domain of (α, β) in the corner layer
$A(x)$	domain of (y, z) in the corner layer
A_C	area of the corner layer
$E(r)$	area of A
$F(r)$	integral of $f\left[\frac{1}{r}\, q\left\{p(\alpha)\, p(\beta)\right\}\right]$ on A
$G(r)$	integral of $f^2\left[\frac{1}{r}\, q\left\{p(\alpha)\, p(\beta)\right\}\right]$ on A
H	dimensionless velocity gradient, $f'(0)$
I_1	integral of $f(\gamma)\,[1-f(\gamma)]$ on $0 \leqslant \gamma \leqslant 1$
I_2	integral of $f(\alpha\beta)\,[1-f(\alpha\beta)]$ on $0 \leqslant \alpha \leqslant 1, 0 \leqslant \beta \leqslant 1$.
I_p	integral of $p(\alpha)$ on $0 \leqslant \alpha \leqslant 1$
J	Jacobian determinant, $\partial(\lambda, \eta)/\partial(\alpha, \beta)$.
K	complete elliptic integral of the first kind
Q'	volumetric rate of crossflow
R	ratio $w(x)/\delta(x)$
U	free-stream velocity

f	dimensionless velocity function
p	dimensionless isovel shape function
q	inverse of p
r	ratio δ (x)/w(x)
s	dummy coordinate in the x-direction
u	x-component of velocity
$w(x)$	width of corner layer
x,y,z	spatial coordinates
α, β	dimensionless coordinates
γ	dimensionless coordinate
λ, η	transformed coordinates
ν	kinematic viscosity of the fluid
δ (x)	thickness of plane layer

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