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Synthesis of Three-Link Function Generators for Coordinated Rotations and Translations

by

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Introduction

Kinematic synthesis is the science and art of designing mechanisms to fulfill a desired set of motion characteristics. The synthesis of mechanisms is classed into three kinds – type synthesis, number synthesis and dimensional synthesis. Type synthesis refers to the kind of machine or mechanism to be used; it may be linkages, belt drives or cams. Number synthesis, on the other hand, deals with determining the number of links, number of joints or pairs to satisfy a required mobility or degree of freedom. The last kind – dimensional synthesis determines the dimensions and orientations of the links in a mechanism.

With regard to the motion characteristic required, there are these three classifications of synthesis problems – *function generation, path generation,* and *motion generation.* Function generation synthesis involves the designs of mechanisms which will transmit the motion of its input link to the output link according to some functional relationship. Path generation synthesis is the process of designing a mechanism wherein one point will describe a desired path. Motion generation synthesis, on the other hand, studies the design of mechanisms that will guide a body through a series of positions and orientations in a plane (in the case of planar mechanisms) or in space (for spatial mechanisms). Motion generation synthesis is also referred to as rigid body guidance.

One of the trends that evolved in the area of spatial mechanisms was towards the study of mechanisms with increasing number of links - five or more. The most commonly studied before were the four-link spatial mechanisms. The mechanisms considered were made up of lower pairs like the revolute (R), the spheric (S), the cylinder (C), and the prismatic (P) pairs. In most cases, the results from their synthesis and analysis were too complex and difficult to apply to many practical problems.

Part of the complexity is due to the high degree of the resulting analysis equations and also due to the difficulty of solving the resulting set of synthesis equations. Many times, deriving the set of synthesis equations is the difficulty – also true for three-link mechanisms.

For the analysis aspect of mechanisms with increased number of links, the analysis equations derived for input-output motion are polynomials of very high

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degrees. The roots of these polynomials give the solutions for the motion parameters. The consequent problem called *branching* is to determine which among the many roots are the true answer(s) or solution(s), because in most cases there is only one *true* solution from the set of roots.

In deriving the input-output motion relation of a mechanism with four or more links, there is also the problem of eliminating the motion parameters of the intermediate link(s). As the number of links are increased, the intermediate link motion parameters are also increased.

Three-Link Space Mechanisms with Higher Pairs

To reduce the effects of the difficulties due to increased number of links, the concept of making use of mechanisms with the least number of links has been proposed.

By making use of higher pairs (pairs with either line or point contact), mechanisms with fewer links can be constructed. The reason for this is that these higher pairs have many degrees of freedom so that they allow more mobility for less number of links. These higher pairs in fact have been represented in the past to be made up of a series of lower pairs from which the analysis has been carried out. Using direct pair geometry constraint equations, the mathematics involved has been reduced.

Only recently has the advantages of using higher pairs been studied extensively. Several papers and studies have been devoted to this class of mechanisms. Among the authors are Singh and Kohli (1), Sandor, Kohli, Hernandez and Ghosal (2), Hernandez, Sandor and Kohli (3), Litvin and Gutnam (4), Ghosal (5) and Hernandez (6). The methods for synthesis of Freudenstein (7), Sandor (8), Chace (9), Suh and Radcliff (10),. Chen and Roth (11), and Kohli and Soni (12) are useful and appropriate.

The number of degrees of freedom of a closed-loop spatial mechanism can be obtained from the following Kutzback equation:

$$\mathrm{DOF} = 6(n-1) - 5(P_1) - 4(P_2) - 3(P_3) - 2(P_4) - (P_5) \tag{1}$$

where n = total number of links

P_i = number of pairs in the mechanism with i degrees of freedom

This equation is obtained from the fact that an n-link mechanism will have (n - 1) moving links (with respect to one fixed link) with a general six degrees of freedom motion, and that each of the P_i kinematic pairs of i degrees of freedom will restrict the motion of the interconnected links by (6 - i) degrees of freedom.

For a single DOF mechanism, n=3 links – 2 links will not make a closed loop – and a dual (rotation and translation) output motion, equation 1 requires that a four DOF pair be used. There are two, four DOF pairs that can be used and these are the cylinder-plane (Cp) and the sphere-groove (Sg) pairs. These two pairs along with a revolute pair and a cylinder pair give us the two three-link function generators. These are the revolute-cylinder-plane-cylinder (R-Cp-C) and the revolute-sphere-groove-cylinder (R-Sg-C) mechanisms – shown in Figures 1 and 2.

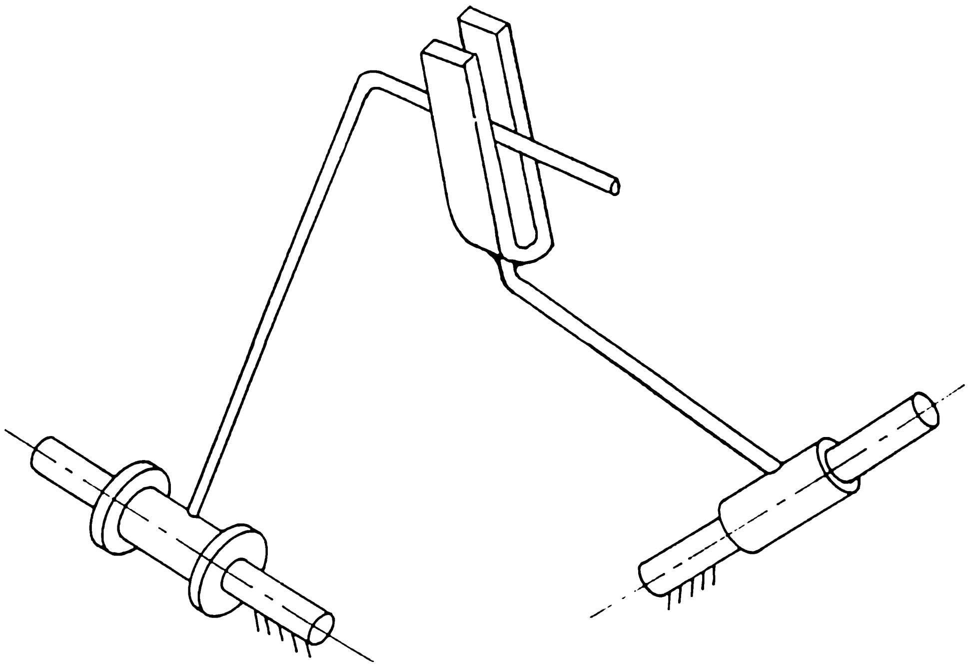

Figure 1. The revolute – cylinder-plane-cylinder (R-Cp-C) mechanism

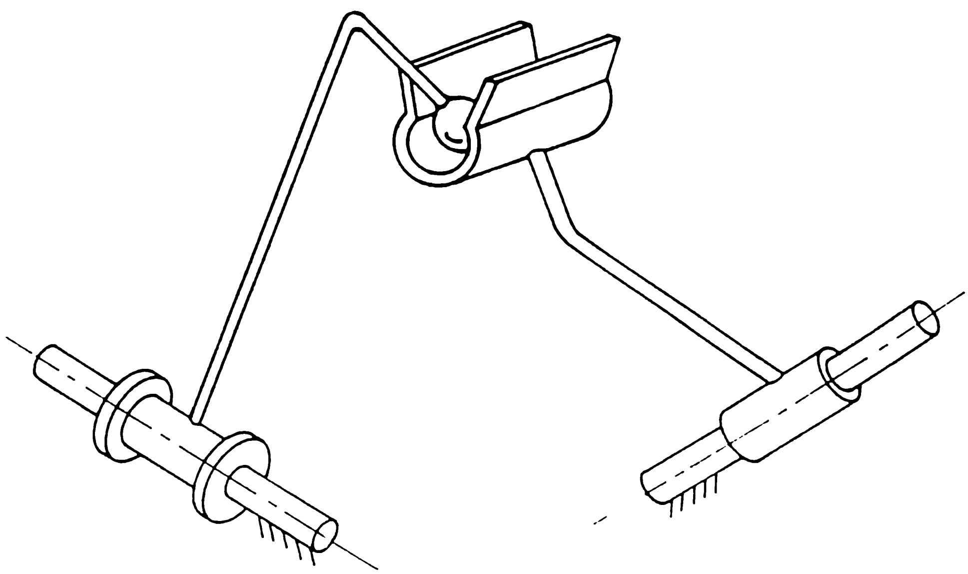

Figure 2. The revolute – sphere-groove – cylinder (R-Sg-C) mechanism

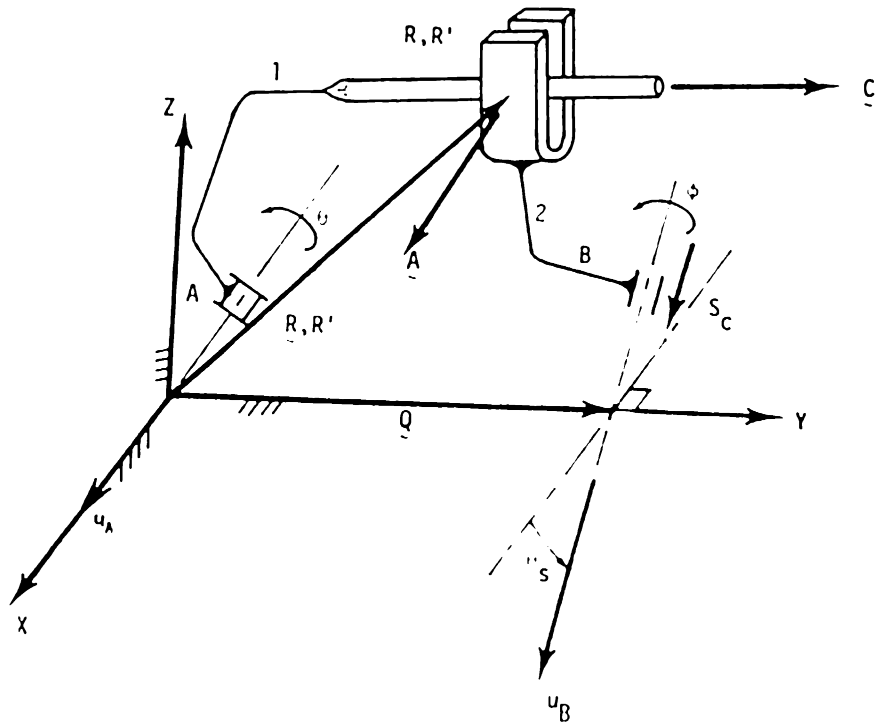

Figure 3. The R-Cp-C mechanism and the associated vectors and scalars

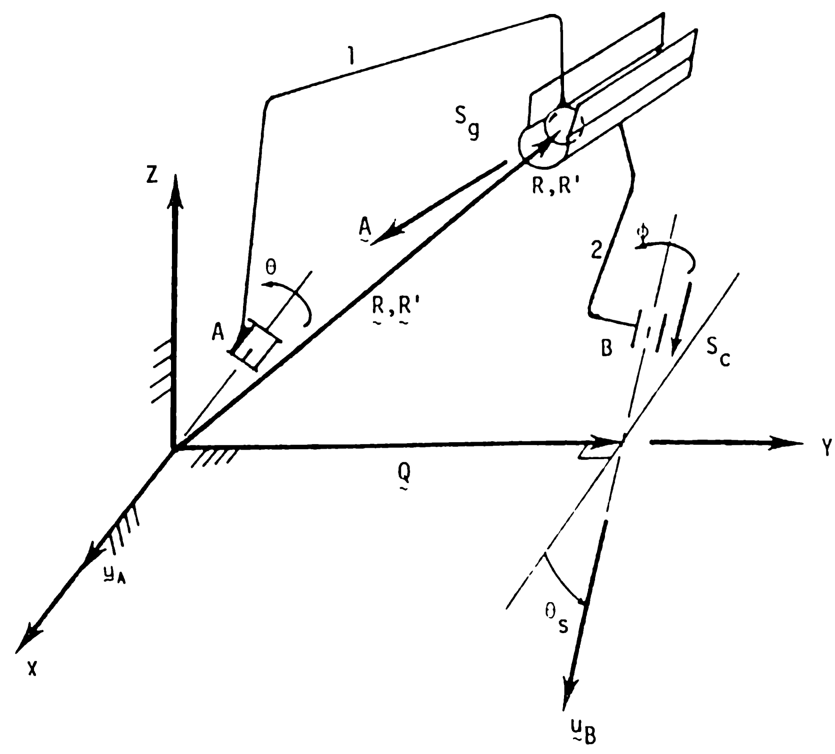

Figure 4. The R-Sg-C mechanism with the associated vectors and scalars

The method for synthesizing these mechanisms to satisfy prescribed motion specifications for the input rotation and output rotation and translation will be shown.

These mechanisms with their associated vectors and scalars are shown in Figures 3 and 4. These vectors and scalars are:

$\underline{u}_A$	–	unit vector defining the direction of the axis of input link 1 at the revolute pair A. This axis is made to coincide with the X-axis such that $\underline{u}_A = 1\hat{i}$
$\underline{u}_B$	–	unit vector defining the axis of the cylinder pair B. It is defined by the skew angle θ_s measured about the common perpendicular of $\underline{u}_A$ and $\underline{u}_B$, which is the Y-axis. Thus, $\underline{u}_B = \cos\ \theta_s i - \sin\ \theta_s k$.
$\underline{Q}$	–	vector along the Y-axis, the common perpendicular of $\underline{u}_A$ and $\underline{u}_B$. $/\underline{Q}/ = 1$ to fix the scale of the mechanism.
$\underline{R}$	–	vector from the origin to the point R in the cylinder axis for the Cp pair, or to the point R the sphere center in the Sg pair.
$\underline{R}'$	–	vector from the origin to the point R', a point fixed to link 2 and initially coincident with R. At the starting position then, $\underline{R} = \underline{R}'$
$\underline{A}$	–	a unit vector fixed to link 2 and perpendicular to the plane in the Cp pair, or a unit vector along the groove axis in the Sg pair.
$\underline{C}$	–	vector along the cylinder axis in the Cp pair.
θ	–	rotation of link 1 about $\underline{u}_A$.
φ	–	rotation of link 2 about $\underline{u}_B$.
S_c	–	linear translation of link 2 along $\underline{u}_B$.

Function generation synthesis of these mechanisms means that the input and output motions (i.e., θ_j, φ_j and S_{cj} and even their derivatives for different jth positions) will be specified with the objective of determining the vectors that describe the mechanism which will fulfill these prescribed motion specifications. As Freudenstein (7) showed in approximate synthesis of mechanisms, we can only specify these motion specifications at particular points in the function requirements. The number of points – called precision points because the mechanism will satisfy exactly these points – is limited by the parameters available for synthesis, the nature of the synthesis equations and also in some cases the degree of difficulty in solving the resulting set of synthesis equations.

Parameters Available for Synthesis

The vectors shown in Figures 3 and 4 are classified into position vectors or design vectors. The position vectors $\underline{u}_A$ and $\underline{Q}$ because they simply locate and orient the three-link mechanisms within a spatial coordinate system. On the other hand, the vectors $\underline{R}$, $\underline{A}$, $\underline{u}_B$ and $\underline{C}$ are design vectors since they define the geometry of the mechanism. A set of these design vectors define a unique mechanism.

For function generation synthesis, the vector $\underline{u}_B$ defining the axis of the output pair is generally specified.

For the R-Cp-C mechanism, the design vectors are $\underline{R}$, $\underline{A}$, and $\underline{C}$. However, the vector $\underline{R}$ in this case locates a vector, $\underline{C}$, in space so that only two of the three parameters of $\underline{R}$ are independent. Thus, since $\underline{A}$ and $\underline{C}$ are unit vectors, the R-Cp-C mechanism is defined by six design parameters – two of the three coordinates of $\underline{R}$ and two unit vectors parameters each for $\underline{A}$ and $\underline{C}$.

For the R-Sg-C mechanism, the design vectors are $\underline{R}$ and $\underline{A}$. This gives five parameters available for design since $\underline{R}$ in this case locates a point in space.

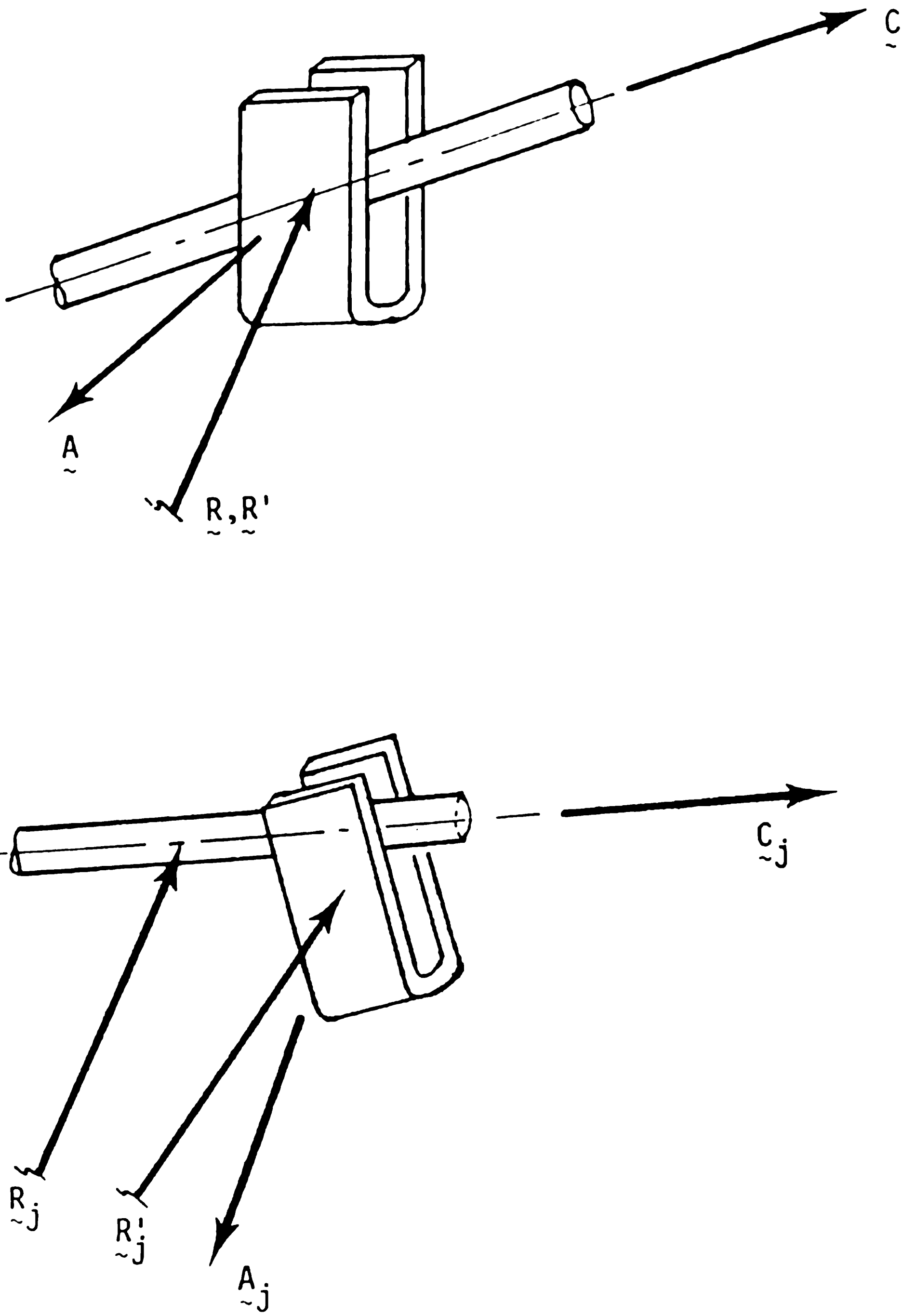

Figure 5. Initial and (jth) positions of the Cp pair

Pair Geometry Constraint Equations

Figure 5 shows the Cp pair in the initial and jth position. The motion between the cylinder and the plane is characterized by two translations of the cylinder with respect to the plane plus two rotations of the cylinder about axis $\underline{A}$ and axis $\underline{C}$. Thus, the relative displacement between points R and R' must be perpendicular to the vector $\underline{A}$ at any displaced position and that the vectors $\underline{A}$ and $\underline{C}$ must always be perpendicular to each other. Expressed as two separate equations, the pair constraint equations are :

$$(R_j - R'_j) \cdot A_j = 0 \tag{2}$$

and

$$\underline{A}_j \cdot \underline{C}_j = 0 \tag{3}$$

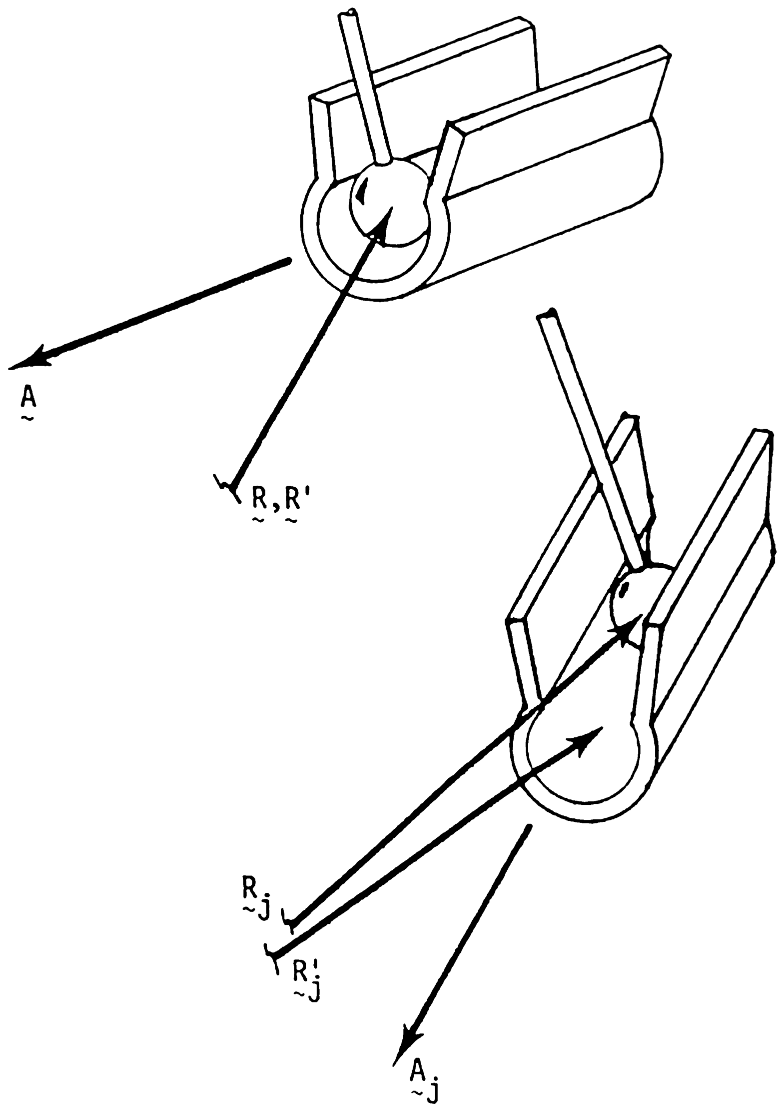

Figure 6. Initial and (jth) positions of the Sg pair

Figure 6 shows the Sg pair at an initial position and at a jth position. The motion of the sphere relative to the groove is made up of a linear translation in the groove and three rotations of the sphere. The relative displacement of points R and R' in this pair must be in the direction of the groove. Written in an equation form, this constrained motion is given by:

$$\underline{R}_j - \underline{R}'_j = S_{gj}\underline{A}_j \tag{4}$$

Vector Displacements and Equations of Motion

Screw displacements in vector form are used to obtain the following expressions. The reader can look up reference (6) for the details of the derivations that will follow. A summarized form is presented here. The following expressions to be used in the pair geometry constraint equations are derived as follows in terms of the motion specifications and the design vectors.

$$\underline{A}_j = \underline{A} + (\cos\varphi_j - 1)\underline{U}_{BA} + \sin\varphi_j\,(\underline{u}_B \times \underline{A}) \tag{5}$$

$$\underline{R}_j = \underline{R} + (\cos\theta_j - 1)\,\underline{U}_{AR} + \sin\theta_j(\underline{u}_A \times \underline{R}) \tag{6}$$

$$\underline{R}'_j = \underline{R} + (\cos\varphi_j - 1)\underline{U}_{BL} + \sin\varphi_j(\underline{u}_B \times \underline{L}) + S_{cj}\underline{u}_B \tag{7}$$

$$\underline{C}_j = \underline{C} + (\cos\theta_j - 1)\underline{U}_{AC} + \sin\theta_j(\underline{u}_A \times \underline{C}) \tag{8}$$

where

$$\underline{U}_{BA} = (\underline{u}_B \times \underline{A}) \times \underline{u}_B\ , \tag{9}$$

$$\underline{U}_{AR} = (\underline{u}_A \times \underline{R}) \times \underline{u}_A\ ,\ \text{etc.} \tag{10}$$

and $\underline{L} = \underline{R} - \underline{Q}$ (11)

Substituting equations 5 to 11 into the pair constraint equations will give us the following equations of motion.

For the R-Cp-C mechanism,

$$[\underline{A} + C_1\varphi_j\underline{U}_{BA} + S\varphi_j\,(\underline{u}_B \times \underline{A})] \cdot [\underline{C} + C_1\theta_j\underline{U}_{AC} + S\theta_j\,(\underline{u}_A \times \underline{C})]$$
$$= 0 \tag{12}$$

and

$$[\, C_1\theta_j \underline{U}_{AR} + S\theta_j(\underline{u}_A \times \underline{R}) - C_1\varphi_j \underline{U}_{BL} - S\varphi_j(\underline{u}_B \times \underline{L}) - S_{cj}\underline{u}_B] \cdot$$

$$[\underline{A} + C_1\varphi_j \underline{U}_{BA} + S\varphi_j (\underline{u}_B \times \underline{A})] = 0 \qquad (13)$$

where $C_1\varphi_j = \cos\varphi_j - 1$ and $S\varphi_j = \sin\varphi_j$

For the R-Sg-C mechanism.

$$C_1\theta_j \underline{U}_{AR} + S\theta_j (\underline{u}_A + \underline{R}) - C_1\varphi_j \underline{U}_{BL} - S\varphi_j (\underline{u}_B \times \underline{L}) - S_{cj}\underline{u}_B$$

$$= S_{gj}[\underline{A} + C_1\varphi_j \underline{U}_{BA} + S\varphi_j(\underline{u}_B \times \underline{A})] \qquad (14)$$

where S_{gj} = motion of the sphere along the groove.

These equations of motions are displacement equations. To obtain higher order equations of motion, the time derivatives of these displacement equations must be obtained. The synthesis equations that will be obtained also allow us to specify not only displacements but also higher order motion specifications.

Number of Positions Available for Synthesis

Although the form and number of constraint equations are different for the two mechanisms considered, both of them can be synthesized for a maximum number of three positions.

For the R-Cp-C mechanism, there are six design parameters available for synthesis. Equations 12 and 13 are to be written together at any jth displaced position from the initial position. In addition to this, Equation 12 must also be satisfied at the initial position, i.e.,

$$\underline{A} \cdot \underline{C} = 0$$

For every displaced position, two scalar equations are written plus the single initial position equation. Thus, we can synthesize the R-Cp-C mechanism for three positions.

For the R-Sg-C mechanism, there are five parameters available for synthesis. The constraint equation is a vector equation and is made up of three scalar equations that are written for each displaced position. However, the set of three scalar equations has the additional unknown S_{gj} for each displaced position. Thus, we also get three positions available for synthesis.

The Synthesis Equations

Carrying out the vector operations and simplifying the results will give us the following synthesis equations for the R-Cp-C mechanism.

$$e_1 d_1 + e_2 d_2 + 1 = 0 \tag{15}$$

$$L_{1j}^{n} + L_{2j}^{n} e_1 + L_{3j}^{n} e_2 + L_{4j}^{n} d_1 + L_{5j}^{n} d_1 e_1 + L_{6j}^{n} d_1 e_2 + L_{7j}^{n} d_2 +$$

$$L_{8j}^{n} d_2 e_1 + L_{9j}^{n} d_2 e_2 = 0 \qquad j = 2, 3 \tag{16}$$

and

$$[(1 + pd_2)M_{1j}^{n} + M_{2j}^{n} d_1] r_1 + [M_{3j}^{n} + M_{4j}^{n} d_1 + M_{5j}^{n} d_2] r_2 +$$

$$[M_{6j}^{n} + M_{7j}^{n} d_1 + M_{8j}^{n} d_2] r_3 + [M_{9j}^{n} + \delta M_{1j}^{n} d_1 + M_{10j}^{n} d_2] = 0$$

$$j = 2, 3 \tag{17}$$

where

$$d_1 = a_2/a_1 \qquad P = \tan\theta_s$$

$$d_2 = a_3/a_1 \qquad \delta = -1/\sin^2\theta s \tag{18}$$

$$e_1 = c_2/c_1$$

$$e_2 = c_3/c_1$$

and a_i, c_i, r_i $(i = 1, 2, 3)$ are the coordinates of the vectors $\underline{A}$, $\underline{C}$ and $\underline{R}$ respectively.

The L_{ij}^{n}'s and M_{ij}^{n}'s are deterministic functions of the specified motion specifications and are derived and listed in reference (6). The superscript n refers to the order of the motion specification, i.e., n = 0 for displacements, n = 1 for velocities and so on. The subscript j refers to the position with j = 1 denoting the initial position.

For the R-Sg-C mechanism, the constraint equation is in vector form and this gives us the synthesis equation written in matrix form as

$$[G_{ikj}] \ \{r_i\} + S_{gj}[H_{ikj}] \ \{a_i\} = \{P_{ij}\} \tag{19}$$

$$i = 1, 2, 3$$

$$j = 2, 3$$

For application to MSP synthesis, the time derivatives of equation 19 is obtained. The elements of the matrices for [G] and [H] can be found in reference (6).

Solutions to Some Synthesis Cases

The solutions to the different synthesis cases, because of the non-linearity of the equations, are best solved using the elimination method. Please see references (13) and (14).

The three-position synthesis case for the R-Cp-C mechanism with unknowns e_1, r_1, r_2, d_1 and d_2 will be considered.

The following equations are obtained from the synthesis equations:

$$e_1 d_1 + e_2 d_2 + 1 = 0 \tag{20}$$

$$(A_j + B_j e_1) d_1 + (C_j + D_j e_1) d_2 + (E_j + F_j e_1) = 0$$
$$j = 2, 3 \tag{21}$$

and
$$(G_j + H_j d_1 + I_j d_2) r_1 + (J_j + K_j d_1 + L_j d_2) r_2$$
$$+ (M_j + N_j d_1 + 0_j d_2) = 0 \qquad j = 2, 3 \tag{22}$$

where the A_j, B_j, C_j .... 0_j coefficients are obtained from the L_{ij}^{n}'s, M_{ij}^{n}'s, and the specified parameters.

The system of five equations can be solved by considering the three equations of (20) and (21) separately from the two equations of (22). In the first set, the solutions for e_1 are obtained from the eliminant, expressed as the determinant,

$$\begin{vmatrix} e_1 & e_2 & 1 \\ (A_2 + B_2 e_1) & (C_2 + D_2 e_1) & (E_2 + F_2 e_1) \\ (A_3 + B_3 e_1) & (C_3 + D_3 e_1) & (E_3 + F_3 e_1) \end{vmatrix} = 0 \tag{23}$$

The one or three real roots of e_1 from the third degree eleminant polynomial are then substituted into any two equations of (20) and (21) to get d_1 and d_2. With d_1 and d_2 obtained, r_1 and r_2 are solved for from the two equations of (22).

The solutions to the other synthesis cases for the two mechanisms including those with higher order motion specifications can be seen in reference (6).

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