"the fastest method of solving these differential waterhammer equations is by digital computer."

Waterhammer Analysis in Pump Discharge Lines Caused By Power Failure*

by

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Abstract

The safe design of a pumping system requires a knowledge of the maximum pump or motor speed, the minimum and maximum pressures that may occur in the system during transient conditions specifically after power failure. Knowledge of the time at which the minimum head, maximum head, zero flow, zero pump speed and the maximum reverse flow occurs at the pump can be helpful for the proper design of discharge valve closure time and other waterhammer protection devices. This paper presents the theoretical background of the waterhammer equations and the different methods of solutions to this unsteady flow phenomena that occur in pipelines. It provides the head and discharge at any time after power failure of a specific point along the pipeline for given initial and boundary conditions.

Introduction

In a water supply system where the source of water is groundwater, a pump is needed to lift water to the desired head for storage and distribution purposes. The safe design of a pumping system requires a knowledge of the maximum pump or motor speed, the minimum and maximum pressures that may occur in the system during transient conditions specifically after power failure. Knowledge of the time at which the minimum head, maximum head, zero flow, zero pump speed and the maximum reverse flow occurs at the pump can be helpful for the proper design of discharge valve closure time and other waterhammer protection devices.

This paper presents the theoretical background of the waterhammer equations and the different methods of solutions to this unsteady flow phenomena that occur in pipelines. It provides the head and discharge at any time after power failure of a specific point along the pipeline for given initial and boundary conditions.

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Transient Conditions at Pump and Discharge Line

Waterhammer in pipelines is defined as the change in pressures above or below normal working pressures caused by changes in velocity of flow. These transient pressure and flow conditions, usually accompanied by a hammering-type noise, are usually severe and the pipeline should be designed to withstand positive and negative pressures caused by this rapid deceleration of the pump motors. Figure 1 shows the time history of the pressure, flow and speed changes at a pump installation produced by power failure at the pump motors .

When the power supply to the pump motor is suddenly cut off, the only energy that is left to drive the pump in the forward direction is the kinetic energy

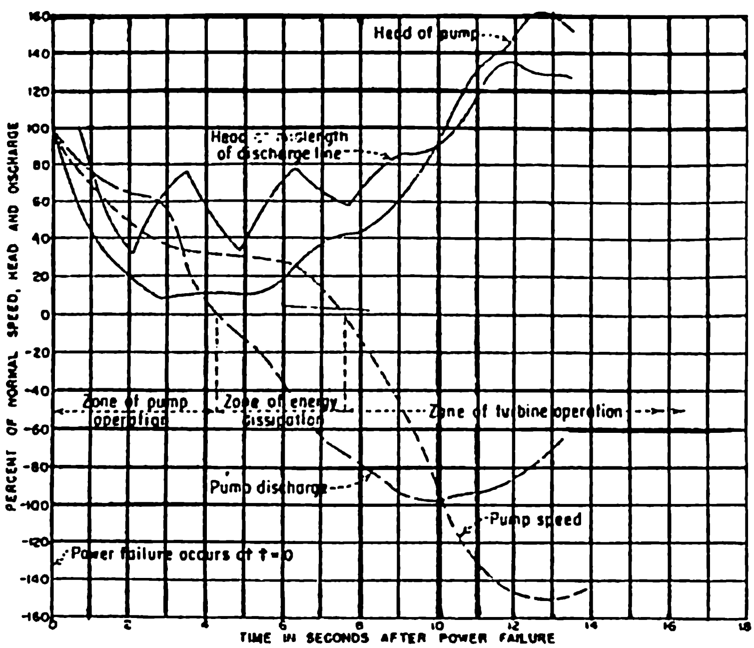


Figure 1. Transient conditions following power failure

of the rotating elements of the motor and pump and the entrained water in the pump. Since this pump inertia is usually small compared to that of the liquid in the discharge line, the reduction in speed is quite rapid. Because the flow and the pumping head at the pump are reduced, negative pressure waves propagate downstream in the discharge line. These subnormal pressure waves move rapidly up the discharge line to the discharge outlet, where a wave reflection occurs. Soon the speed of the pump is reduced to a point where no water can be delivered against the existing head. If there is no control valve present at the pump, the flow through the pump reverses, although the pump may still be rotating in the forward direction. In this condition, the pump is said to be operating in the zone of energy dissipation. Because of the reverse flow, the pump slows down rapidly, stops momentarily, and then reverses, i.e., the pump is now operating as a turbine. The pump speed increases in the reverse direction until it reaches the run away speed. With the increase in the reverse speed, the reverse flow through the pump is reduced due to choking effect, and positive and negative pressure waves are produced in the discharge and suction lines respectively.

If the pipeline profile is such that the transient-state hydraulic grade line falls below the pipeline at any point, vacuum pressure may occur, and the water column in the pipeline may separate at that point. Excessive pressure will be produced when the two columns later rejoin.

Basic Equations for Waterhammer Analysis

In order to determine the transient hydraulic conditions at the pump and discharge line subsequent to a power failure at the pump motors, three effects must be considered; namely, the waterhammer wave equation, the pump and motor inertia and the pump characteristics.

1. Waterhammer wave equation

Chaudhry, Wylie and Watters discuss a detailed theoretical derivation of the basic differential equations for transient flow which are the dynamic equation

$$\frac{\partial H}{\partial x} + \frac{V}{g}\frac{\partial V}{\partial x} + \frac{1}{g}\frac{\partial V}{\partial t} + \frac{fV/V/}{2gD} = 0$$

and the continuity equation

$$\frac{\partial H}{\partial t} + \frac{V\partial H}{\partial x} + \frac{a^2\partial V}{g\ \partial x} - \frac{V\partial z}{\partial x} = 0$$

In these wquations, x is measured from an upstream origin and v is assumed positive if the flow is in the direction of increasing x. See Figure 2.1.

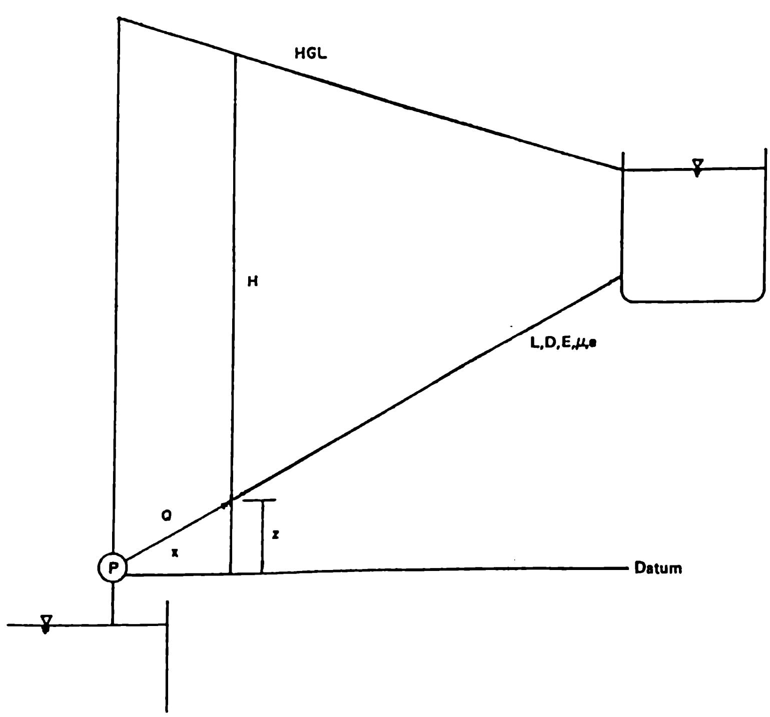


Figure 2.1 Definition Sketch for the Different Parameters used in Waterhammer Equations.

The velocity of propagation of pressure waves in a pipeline filled with liquid is

$$a = \sqrt{\frac{1}{\frac{\gamma}{g}\left(\frac{1}{K} + \frac{Dc}{Ee}\right)}}$$

where values of c are as follows;

$c = \frac{5}{4} - \mu$ for a pipe anchored at the upper end and without expansion joints

$c = 1 - \mu^2$ for a pipe anchored against longitudinal movement throughout its length

$c = 1 - \mu/2$ for a pipe with expansion joints

2. Pump and Motor Inertia

When the power to the pump motor is suddenly cut off, the deceleration of the pump at any instant depends upon the flywheel effect of the rotating parts of the pump and motor and the instantaneous torque, T, exerted by the pump impeller. Hence,

$$T = \frac{-WR^2}{g}\frac{d\omega}{dt} \qquad \text{or} \qquad T = \frac{-WR^2}{g}\frac{2\Pi\, dN}{60\ dt}$$

in which WR^2 is the flywheel effect of rotating parts of motor, pump and entrained water in lb-ft^2, and ω and N are rotational speed, in rad/s and in rpm, respectively. For a small time interval $\Delta t = t_2 - t_1$, this equation is written approximately as follows,

$$\frac{T_1 + T_2}{2} = \frac{2\Pi WR^2 (N_1 - N_2)}{60\ g\ \Delta t}$$

This equation is written with the ratios $a = N/N_R$ and $\beta = T/T_R$ as follows;

$$a_1 - a_2 = \frac{15gT_R(\beta 1 + \beta_2)\Delta t}{\Pi WR^2\ N_R}$$

The decelerating torque at the rated head and pump speed is

$$T_R = \frac{60\ \gamma\ H_R Q_R}{2\Pi\ N_R n_R}$$

in which n_R is the pump efficiency at rated conditions. Then

$$a_1 - a_2 = K_1(\beta_1 + \beta_2)\ \Delta t$$

where

$$K_1 = \frac{450 g\gamma\ H_R Q_R}{\Pi^2 WR^2 \eta_R N_R}$$

3. Pump Characteristics

The discharge, Q, and the pressure head, H, at the boundary must be known in order to develop the boundary conditions. The discharge of a centrifugal pump depends upon the rotational speed, N, and the pumping head, H; and the transient-state speed changes depend upon torque, T, and the combined moment of inertia of the pump, motor, and the liquid entrained in the pump impeller. Thus, four variables – namely Q, H, N, and T – have to be specified for the mathematical representation of a pump. The curves showing the relationships between these variables are called the pump characteristics.

Data for prototype pump characteristics are obtained from model test results by using homologous relationships. Two pumps are considered homologous if they are geometrically similar and the streamflow pattern through them is also similar. For homologous pumps, the following ratios are valid

$$\frac{H}{N^2} = \text{constant} \qquad \frac{N}{Q} = \text{constant}$$

These equations may be nondimensionalized by using the quantities for the rated condition as reference values. Let us define the following dimensionless variables:

$$v = Q/Q_R, \quad h = H/H_R, \quad \alpha = N/N_R, \quad \beta = T/T_R$$

Therefore, h/α^2 = constant and α/v = constant

Since α becomes zero while analyzing transients for all four zones of operation, h/α^2 becomes infinite. To avoid this, the parameter $h/(\alpha^2 + v^2)$ instead of h/α^2 may be used.

The signs of v and α depend upon the zones of operation. In addition to the need to define a different characteristic curve for each zone of operation, α/v becomes infinite for $v = 0$. To avoid this a new variable θ may be defined as $\theta = \tan^{-1}\frac{\alpha}{\gamma}$ and then the characteristic curve may be plotted between θ and $h/(\alpha^2 + v^2)$. By definition, θ is always finite, and its value varies between 0° and 360° for the four zones of operation.

Similar to the pressure-head curve, the torque characteristic curve may be plotted between $\beta/(\alpha^2 + v^2)$ and θ.

Using the data presented by Thomas, characteristic curves for pumps having specific speed (specific speed = $N_R\sqrt{Q_R}/H_R^{3/4}$) of 25, 147 and 261 SI units (1276, 7600, and 13500 gpm units, respectively) are presented in Figure 2.2 and in Appendix A.

Methods of Solutions

To make a complete transient analysis on a pump discharge line requires the simultaneous solution of the pump inertia equation and the pressure wave equations in conjunction with the equation for the hydraulic losses in the line and a complete four-quadrant pump characteristic.

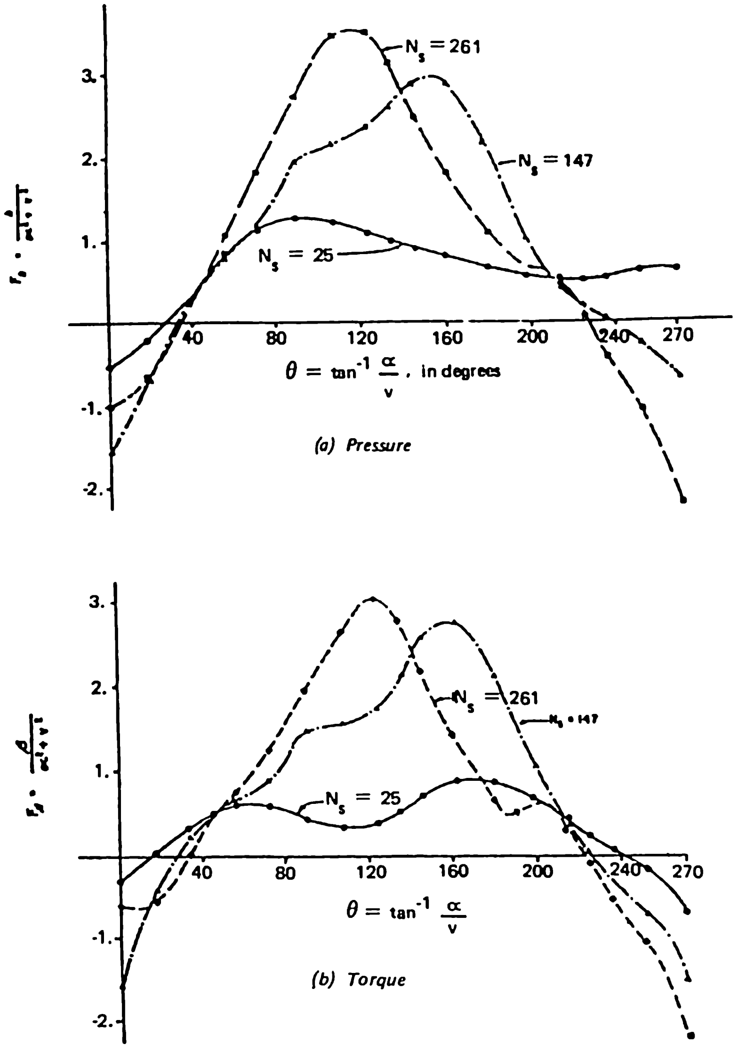


Figure 2.2 Characteristics of pumps of various specific speeds

The dynamic and continuity equations are quasi-linear, hyperbolic, partial differential equations in terms of two dependent variables, velocity and hydraulic-grade-line elevation, and two independent variables, distance along the pipe and time. A closed-form solution of these equations is impossible. However, by neglecting or linearizing the nonlinear terms, various graphical and analytical methods have been developed.

1. In the implicit finite-difference method, the partial derivatives are replaced by finite differences, and the resulting algebraic equations for the whole system are then solved simultaneously. Depending upon the size of the system, this involves a simultaneous solution of a large number of nonlinear equations. The analysis by this method becomes even more complicated in systems having complex boundary conditions, which must be solved by an iterative technique. One of the iterative methods of solving nonlinear simultaneous equations is by Newton-Raphson Method. The method has the advantage that it is unconditionally stable. Therefore, larger time steps can be used, which results in economizing computer time. However the time step cannot be increased arbitrarily because it results in smoothing the pressure peaks.

2. In the method of characteristics, the partial differential equations are first converted into ordinary differential equations which are then solved by an explicit finite-difference technique. Because each boundary condition and each conduit section are analyzed separately during a time step, this method is particularly suitable for systems with complex boundary conditions. The disadvantage of this method is that small time steps must be used to satisfy the Courant condition for stability. A brief derivation of the characteristic equations is given below.

Rewriting the dynamic and continuity equations in simplified form, we have

$$L_1 = \frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + \frac{fQ|Q|}{2DA} = 0 \qquad 3.1$$

$$L_2 = a^2\frac{\partial Q}{\partial x} + gA\frac{\partial H}{\partial t} = 0 \qquad 3.2$$

Let us consider a linear combination of Equations 3.1 and 3.2, i.e.,

$$L = L_1 + \lambda L_2$$

or

$$\left(\frac{\partial Q}{\partial t} + \frac{\lambda a^2 \partial Q}{\partial x}\right) + \lambda gA\left(\frac{\partial H}{\partial t} + \frac{1}{\lambda}\frac{\partial H}{\partial x}\right) + \frac{f}{2DA}Q|Q| = 0 \qquad 3.3$$

If $H = H(x,t)$ and $Q = Q(x,t)$ are solutions of Equations 3.1 and 3.2, then the total derivatives may be written as

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\frac{dx}{dt} \qquad 3.4$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{dx}{dt} \qquad 3.5$$

By defining the unknown multiplier λ as

$$\frac{1}{\lambda} = \frac{dx}{dt} = \lambda a^2 \qquad 3.6$$

or

$$\lambda = \pm\frac{1}{a} \qquad 3.7$$

and by using Eqs. 3.4 and 3.5, Eq. 3.3 can be written as

$$C+ : \frac{dQ}{dt} + \frac{gAdH}{a\,dt} + \frac{fQ|Q|}{2DA} = 0 \qquad 3.8$$

if

$$\frac{dx}{dt} = a \qquad 3.9$$

and $C- \quad : \frac{dQ}{dt} - \frac{gAdH}{a \ dt} + \frac{f \ Q|Q|}{2DA} = 0$ 3.10

if $\frac{dx}{dt} = -a$ 3.11

Hence, by imposing the relations given by Eqs.3.9 and 3.11, we have converted the partial differential equations into ordinary differential equations in the independent variable t. In the x-t plane, Eqs. 3.9 and 3.11 represent two straight lines having slopes $\pm 1/a$. These are called characteristic lines.

To solve Eqs. 3.8 through 3.11, a number of finite difference schemes have been proposed: Streeter and Wylie use a first-order finite-difference technique; Evangelisti suggests a predictor-corrector method; and Lister employs both first and second order finite-difference scheme. Because the time interval used in solving these equations for practical problems are usually small, a first order technique is sufficiently accurate. However, if the friction losses are large, then a first-order approximation may yield unstable results. For such cases, a predictor–corrector method or a second-order approximation should be used to avoid instability of the finite-difference scheme. In a second order approximation, an average value of the friction term computed at points P and A (Figure 3.1) is used for Equation 3.8, and an average value of the friction term computed at points P and B is used for Equation 3.10. This results in two nonlinear algebraic equations in Q_p and H_p which may be solved by the Newton – Raphson Method. In the predictor-corrector scheme, a first-order approximation is used to determine the discharge at the

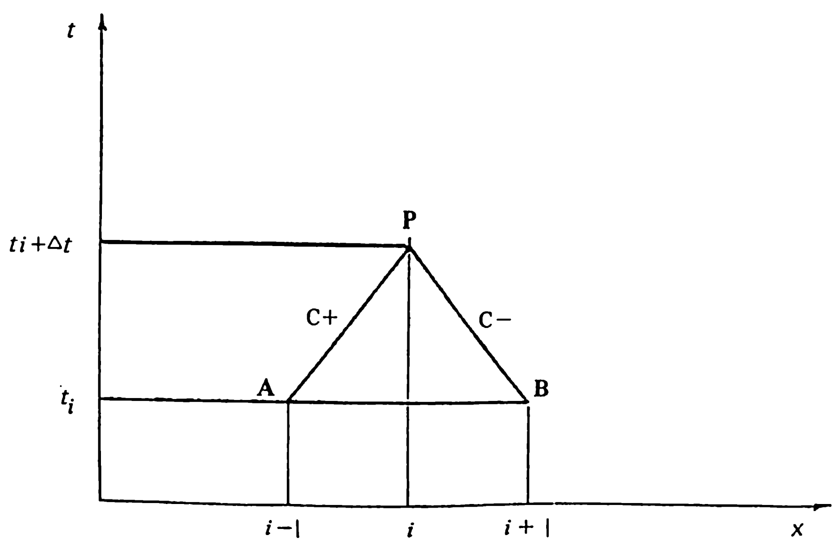


Figure 3.1 Characteristic lines in the x - t plane

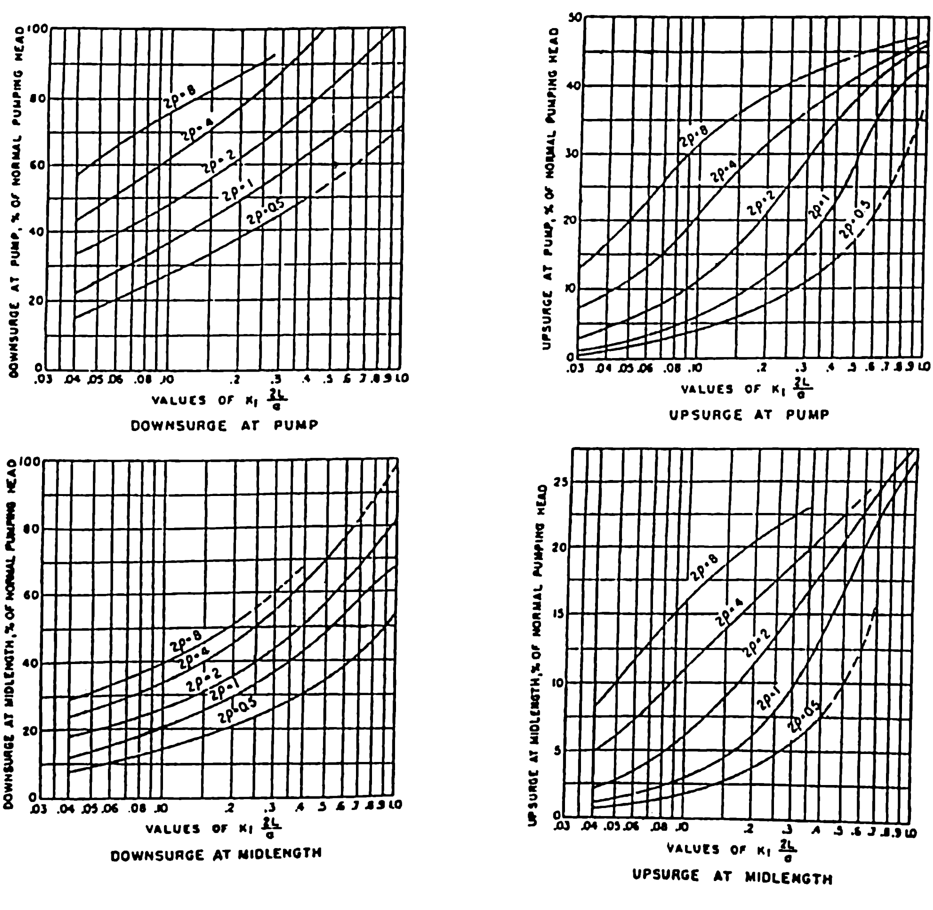
$$C+: \ (Q_p - Q_A) + \frac{gA}{a}(H_p - H_A) + \frac{f\Delta t}{2DA} Q_A |Q_A| = 0$$

$$C-: \ (Q_p - Q_B) - \frac{gA}{a}(H_p - H_B) + \frac{f\Delta t}{2DA} Q_B |Q_B| = 0$$

end of the time step. This predicted value of the discharge is then used in the corrector part to compute the friction term.

The fastest method of solving these differential waterhammer equations is by digital computer. The conduit is divided into a number of intervals and Δt is set equal to $\Delta x/a$. Chaudhry, Wylie and Watters discuss numerical methods of waterhammer analysis and they provide computer programs for the solution of this unsteady flow problem. Fox also tackles hydraulic analysis of unsteady flow in pipe networks. He gives detailed attention to the methods of dealing with boundary conditions such as reservoirs, junctions, valves, air vessels and surge tanks as well as a variety of pumps.

3. A common method of analysis for waterhammer pressures used to be the graphical method. Friction was assumed to be concentrated at one end of the pipe or at a few points along the line, and the waterhammer equations were solved simultaneously with the pump characteristics on a graph of $h = H/H_0$ plotted against $v = V/V_0$ for successive time intervals. Parmakian presents a brief and comprehensive discussion on graphical waterhammer analysis. In addition, he provides waterhammer charts which furnish a convenient method for obtaining the limiting transient conditions at the pump and discharge line when no control valves are present at the pump. See Figure 3.2. Bergeron and Pickford also discuss graphical methods of solving waterhammer equations.



$$p = \frac{aV_0}{2gH_R}$$

Figure 3.2 Waterhammer Charts

APPENDIX A

PUMP CHARACTERISTIC DATA*

$\theta = \tan^{-1}\frac{\alpha}{v}$ (Degrees)	$N_s = 25$		$N_s = 147$		$N_s = 261$	
	$\frac{h}{\alpha^2 + v^2}$	$\frac{\beta}{\alpha^2 + v^2}$	$\frac{h}{\alpha^2 + v^2}$	$\frac{\beta}{\alpha^2 + v^2}$	$\frac{h}{\alpha^2 + v^2}$	$\frac{\beta}{\alpha^2 + v^2}$
0	- 0.530	- 0.350	- 1.560	- 1.560	- 1.000	- 0.560
5	- 0.476	- 0.474	- 1.290	- 1.200	- 0.948	- 0.600
10	- 0.392	- 0.180	- 1.035	- 0.895	- 0.892	- 0.605
15	- 0.291	- 0.062	- 0.795	- 0.600	- 0.820	- 0.580
20	- 0.150	0.037	- 0.540	- 0.355	- 0.665	- 0.503
25	- 0.037	0.135	- 0.308	- 0.135	- 0.475	- 0.355
30	0.075	0.228	- 0.082	0.060	- 0.275	- 0.160
35	0.200	0.320	+ 0.122	0.235	- 0.055	+ 0.070
40	0.345	0.425	0.310	0.380	+ 0.200	0.320
45	0.500	0.500	0.500	0.500	0.500	0.500
50	0.655	0.548	0.635	0.580	0.785	0.620
55	0.777	0.588	0.745	0.645	1.035	0.708
60	0.900	0.612	0.860	0.695	1.280	0.825
65	1.007	0.615	0.982	0.755	1.508	0.955
70	1.115	0.600	1.140	0.850	1.730	1.150
75	1.188	0.569	1.365	0.970	1.970	1.413
80	1.245	0.530	1.595	1.115	2.225	1.608
85	1.278	0.479	1.790	1.300	2.485	1.780
90	1.290	0.440	1.960	1.485	2.740	1.960
95	1.287	0.402	2.048	1.518	2.980	2.150
100	1.269	0.373	2.110	1.540	3.195	2.345
105	1.240	0.350	2.158	1.545	3.380	2.525
110	1.201	0.340	2.203	1.560	3.515	2.710
115	1.162	0.340	2.250	1.592	3.572	2.900
120	1.115	0.350	2.315	1.642	3.570	3.000
125	1.069	0.380	2.390	1.720	3.480	3.010
130	1.025	0.437	2.495	1.900	3.350	2.925
135	0.992	0.520	2.630	2.090	3.140	2.760
140	0.945	0.605	2.785	2.315	2.875	2.500
145	0.908	0.683	2.905	2.530	2.570	2.245
150	0.875	0.750	3.000	2.650	2.300	1.990
155	0.848	0.802	3.020	2.720	2.065	1.750
160	0.819	0.845	2.975	2.740	1.840	1.518
165	0.788	0.872	2.825	2.685	1.633	1.300
170	0.755	0.883	2.652	2.535	1.440	1.085
175	0.723	0.878	2.442	2.310	1.260	0.870
180	0.690	0.860	2.185	2.090	1.080	0.660
185	0.656	0.823	1.890	1.850	0.920	0.500
190	0.619	0.780	1.525	1.570	0.780	0.505
195	0.583	0.725	1.195	1.250	0.710	0.555
200	0.555	0.660	0.935	0.955	0.670	0.615
205	0.531	0.580	0.695	0.730	0.660	0.630
210	0.510	0.490	0.500	0.530	0.555	0.500
215	0.502	0.397	0.374	0.350	0.410	0.315
220	0.500	0.310	0.277	0.175	0.265	0.100
225	0.505	0.230	0.190	0.000	0.065	- 0.075
230	0.520	0.155	0.114	- 0.160	- 0.140	- 0.315
235	0.539	0.085	0.058	- 0.295	- 0.345	- 0.515
240	0.565	0.018	- 0.015	- 0.425	- 0.550	- 0.715
245	0.593	- 0.052	- 0.110	- 0.550	- 0.745	- 0.880
250	0.615	- 0.123	- 0.220	- 0.670	- 0.960	- 1.030
255	0.634	- 0.220	- 0.334	- 0.820	- 1.200	- 1.225
260	0.640	- 0.348	- 0.440	- 0.992	- 1.480	- 1.450
265	0.638	- 0.490	- 0.550	- 1.213	- 1.810	- 1.860
270	0.630	- 0.680	- 0.670	- 1.500	- 2.200	- 2.200

*These pump characteristic data are based on data presented by Thomas and Donsky.

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NOTATION

SYMBOL	DESCRIPTION
1. Pipe Dimensions and General Symbols	
A	cross-sectional area of pipe
e	thickness of pipe wall
D	inside diameter of pipe
E	modulus of elasticity of pipe wall material
μ	Poison's ratio for the pipe wall material
L	length of pipe
x	distance measured from an upstream origin
t	time
z	elevation of pipe above a reference datum
γ	specific weight of water
g	acceleration due to gravity
K	Bulk modulus of water
H_o	pressure head for steady conditions
H	pressure head for unsteady conditions

h	H/H_o
V_o	velocity in pipe for steady conditions
Q_o	flow in pipe for steady conditions
V	velocity in pipe for unsteady conditions
Q	flow in pipe for unsteady conditions
v	V/V_o
f	pipe friction factor
a	velocity of pressure wave

2. Pump Operation Symbols

N	pump speed in rpm
N_R	rated pump speed
H_R	rated pumping head
T	instantaneous torque
T_R	torque at rated head and pump speed
a	N/NR
β	T/TR
η	pump efficiency at rated speed and head
ω	angular velocity of pump and motor shaft
WR^2	flywheel effect of rotating parts of motor, pump and entrained water
θ	$\tan^{-1} \frac{a}{v}$

3. Air Chamber Symbols

C	volume of compressed air in air chamber
c	ratio of volume of compressed air in air chamber at any time to initial volume, $c = C/C_o$
H^*	pressure head at air chamber at any time referred to absolute zero
h^*	ratio of absolute pressure head at air chamber at any time to initial pressure head, $h^* = H^*/H_o^*$
ρ^*	$aV_o/(2g\ H_o^*)$ – pipeline characteristic
K	coefficient of head loss such that KH_o^* is the total head loss for a flow of Q_o into air chamber