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Kinematic Analysis of Three-Link Spatial Mechanisms Containing Sphere-Plane and Sphere-Groove Pairs *

by:

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Abstract

Kinematic pairs in a spatial mechanism are viewed either as allowing relative screw motion between links or as constraining the motion of the two chains of the mechanism connected to the two elements of the pair. Using pair geometry constraints of the sphere-plane and sphere-groove kinematic pairs, the displacement, velocity and acceleration equations are derived for, R-Sp-R,* R-Sp-P, P-Sp-P, P-Sp-R and R-Sg-C three-link mechanisms. For known values of the input variable, other variables are computed in closed form. The analysis procedures are illustrated using numerical examples.

Introduction

The mechanisms containing higher pairs such as cams, sphere-plane, sphere-groove, or cylinder-plane provide the designer with the capabilities of designing machines and mechanisms to satisfy more complex and exact functional requirements than feasible with only lower pair mechanisms. These mechanisms in general are compact and contain fewer links than those with lower pairs.

In recent years, there has been considerable development in the tools for kinematic analysis of spatial mechanisms containing lower pairs.

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Kinematic analysis of space mechanisms was initiated by the significant contribution of Dimentberg¹. Dimentberg^{2,3} demonstrated the use of dual numbers and screw calculus to obtain closed-form displacement relationships of an RCCC* and other four-, five-, six-, and seven-link spatial mechanisms containing revolute, cylinder, prismatic and helical pairs. Denavit⁴ derived closed-form displacement relationships for a spatial RCCC mechanism using dual Euler angles. Yang⁵ also derived such relationships for RCCC mechanisms using dual quaternions.

Vectors were first used by Chace⁶ to derive closed-form displacement relations of RCCC mechanisms. Wallace and Freudenstein⁷ also used vectors to obtain closed-form displacement relations of RRSRR and RRP_LRR mechanisms.

Yang⁸ proposed a general formulation using dual numbers to conduct displacement analysis of RCRCR spatial five-link mechanisms. Soni and Pamidi⁹ extended this application of (3 x 3) matrices with dual elements to obtain closed-form displacement relations of RCCRR mechanisms.

Yuan¹⁰ employed screw coordinates to obtain closed-form displacement relations for RRCCR and other spatial mechanisms.

Jenkins and Crossley¹¹, Sharma and Torfason¹², Dukkupati and Soni¹³ used the method of generated surfaces to conduct analysis of single loop mechanisms containing revolute, prismatic, cylinder, helical and spheric pairs. Hertenberg and Denavit¹⁴ contributed iterative techniques to conduct displacement analysis of spatial mechanisms using (4 x 4) matrices containing revolute, prismatic, cylinder, helical and spheric pairs. Uicker¹⁵ explored in further detail the (4 x 4) matrix approach of Hartenberg and Denavit. Soni and Harrisberger¹⁶ contributed an iterative approach for performing kinematic analysis using (3 x 3) matrices with dual elements. Kohli and Soni^{17,18} used finite screws to conduct displacement analysis of single-loop and two-loop space mechanisms involving R, P, C, H and S pairs.

Bagci¹⁹ used a (3 x 3) screw matrix for displacement analysis of a mechanism containing two revolute pairs, one cylinder pair and one spheric pair. Dobrovolski²⁰ used the method of spherical images to analyze space mechanisms containing revolute and cylinder pairs. Duffy^{21,22}, Duffy and Habib-Olahi²³ used the method of spherical triangles to derive displacement relations for five and six link mechanisms containing revolute and cylinder pairs. Keller²⁵ and Gupta²⁶ also analyzed space mechanisms containing

*R: revolute, P: prismatic, C: cylindrical, S: spherical, Sp: sphere-plane and Sg: sphere-groove joint.

revolute, prismatic, cylinder, helical, and spheric pairs. Recently Kohli and Soni²⁶ and Singh and Kohli²⁷ used the method of pair constraint geometry and successive screw displacement to conduct analyses of single and multiloop mechanisms.

Since Revolute (R) and Prismatic (P) pairs are special cases of the cylinder pair (in prismatic pairs, the rotation is zero; for revolute pairs sliding is zero), we derive the analysis equation for C-Sp-C and C-Sg-C mechanisms, and then force rotations or translations at one or more pairs to zero, to obtain the equations for the above described three-link one degree of freedom mechanisms.

Briefly, the procedure for obtaining the analysis equations is as follows:

- Step 1. Consider the C-Sp-C mechanism and the C-Sg-C mechanism.
- Step 2. Separate the two moving links (Bodies 1 & 2) at the sphere-plane pair for the C-Sp-C case and at the sphere-groove pair for the C-Sg-C case.
- Step 3. Use the screw displacements in vector form to describe the new (jth) position of the sphere-plane (Sp) or sphere-groove (Sg) pairs from two sides of the pair.
- Step 4. Use the pair geometry constraints on the position of the pair obtained from two sides.
- Step 5. Force the cylindrical (C) joints as revolute (R) or prismatic (P) joints by setting the sliding or the rotation equal to zero at cylindrical pairs.

The Three-link Mechanism and Associated Vectors

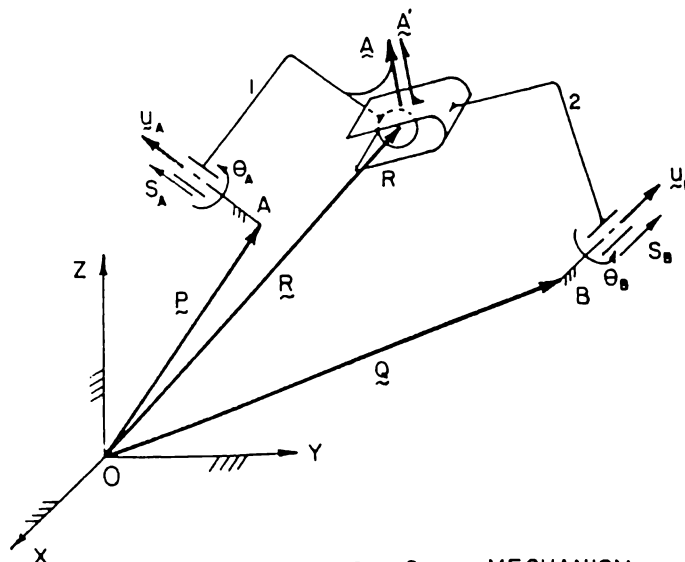


FIGURE 1 C - S_p - C MECHANISM

Figure 1 shows the initial position of two rigid bodies grounded via cylindrical pairs and connected together by a sphere-plane pair. Also shown are the following vectors and scalar quantities:

- \underline{u}_A - unit vector defining the direction of the axis of cylindric pair A.
- \underline{u}_B - unit vector defining the direction of the axis of cylindric pair B.
- \underline{P} - vector locating the axis of cylindric pair at A in the fixed coordinate system.
- \underline{Q} - vector locating the axis of cylindric pair at B in the fixed coordinate system.
- \underline{A} - unit vector perpendicular to the plane of the Sp pair embedded in body 1.
- \underline{A}' - vector embedded in body 2, congruent with \underline{A} in the starting position, as shown in Fig. 1.
- \underline{R} - vector locating point R, the sphere center in the fixed coordinate system.
- θ_A - rotation of link 1 about axis \underline{u}_A .
- θ_B - rotation of link 2 about axis \underline{u}_B .
- S_A - translation of link 1 along axis \underline{u}_A .
- S_B - translation of link 2 along axis \underline{u}_B .

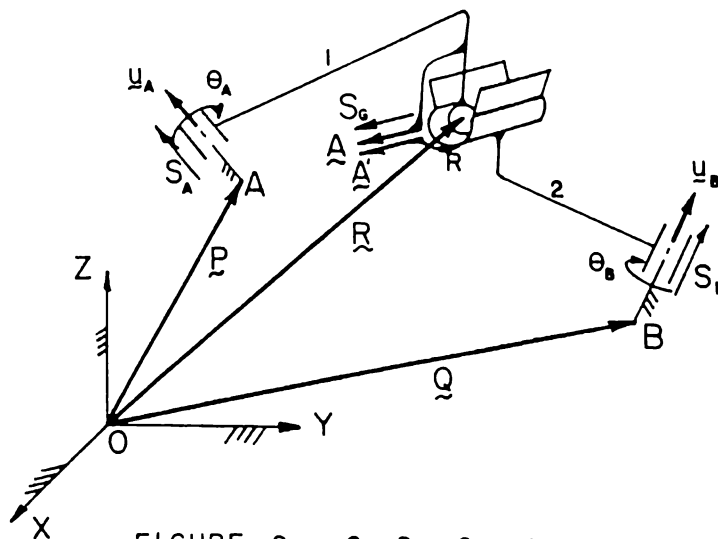


FIGURE 2 C-S_g-C MECHANISM

Figure 2 shows the C-S_g-C mechanism with all associated vectors and scalars. Description of all parameters are the same as for the C-Sp-C mechanism except for the direction of the vector \underline{A} , which is now along the direction of the groove and also the addition of S_G , which is the translation of the sphere along the direction of \underline{A} .

Pair Geometry Constraint Equations

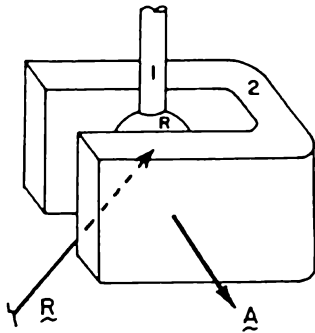


FIGURE 3 SPHERE-PLANE (S_p) PAIR

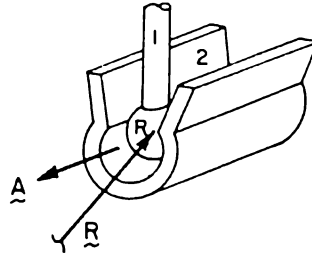


FIGURE 4 SPHERE-GROOVE (S_g) PAIR

Figures 3 and 4 show a sphere-plane (S_p) pair and a sphere-groove (S_g) pair with the vector \underline{R} locating R, the sphere center. The vector \underline{A} , in the S_p pair is defined as a vector perpendicular to the plane in which the sphere moves. In the S_g pair, the vector \underline{A} defines the direction of the groove.

We can now define the vectors \underline{R}_j , \underline{A}_j , \underline{R}'_j and \underline{A}'_j . These new vectors will define the displaced position and direction of initially coincident point R and vector A in bodies 1 and 2 respectively after some relative motion between bodies 1 and 2. The prime notation here is used for new position expressed from the motion of body 2, whereas the unprimed notations are used for new positions expressed from the motion of body 1.

The pair geometry constraint equation for the S_p pair is:*

$$\frac{d^n}{dt^n} (\underline{R}_j - \underline{R}'_j) \cdot \underline{A}'_j = 0, \quad n = 0, 1, 2, \dots \quad (1)$$

which expresses that any relative motion between the sphere and the plane must be perpendicular to the vector \underline{A}'_j (Fig. 1).

The pair geometry constraint equation for the S_g pair is:

$$\frac{d^n \underline{R}_j}{dt^n} - \frac{d^n \underline{R}'_j}{dt^n} = \frac{d^n}{dt^n} (\underline{A}'_j S_{Gj}) \quad n = 0, 1, 2, \dots \quad (2)$$

*See Appendix for the derivation from the complete constraint equation.

where S_{Gj} is the translation of the sphere along the groove in the direction of \underline{A}'_j . The constraint equation for the Sg pair expresses that any relative motion between the sphere and the groove must be along the groove which is in the direction of \underline{A}'_j (Fig. 2).

Working Equations

Referring to Fig. 1, let \underline{A} be a vector in body 1 and \underline{A}' a momentarily congruent vector in body 2 in the first position, perpendicular to the plane of the Sp pair. After some displacement of the mechanism, these vectors, in general, will separate due to the relative motion of the joint elements. Noting that both bodies 1 and 2 are connected to ground by C pairs, we use the equations developed by D. Kohli and A. H. Soni²⁶ for expressing the direction of a vector embedded in the rigid body and also the displaced position of a point of the body after a rotation θ about the cylinder axis and a translation S along the same axis. Using the prime notation for positions of the vector \underline{A}' obtained from the motion of body 2 and the unprimed notation for positions of vector \underline{A} (assumed frozen in body 1 in the first position and then moving with body 1) from the motion of body 1, the displaced directions of the vector \underline{A} in bodies 1 and 2 are:

$$\underline{A}_j = \cos \theta_{Aj} [\underline{A} - (\underline{A} \cdot \underline{u}_A)\underline{u}_A] + \sin \theta_{Aj} (\underline{u}_A \times \underline{A}) + (\underline{A} \cdot \underline{u}_A)\underline{u}_A \quad (3)$$

$$\underline{A}'_j = \cos \theta_{Bj} [\underline{A} - (\underline{A} \cdot \underline{u}_B)\underline{u}_B] + \sin \theta_{Bj} (\underline{u}_B \times \underline{A}) + (\underline{A} \cdot \underline{u}_B)\underline{u}_B \quad (4)$$

Also, the displaced position of the point R in rigid bodies 1 and 2 are given by:

$$\begin{aligned} \underline{R}_j = & \cos \theta_{Aj} [(\underline{R} - \underline{P}) - ((\underline{R} - \underline{P}) \cdot \underline{u}_A)\underline{u}_A] + \sin \theta_{Aj} (\underline{u}_A \times (\underline{R} - \underline{P})) \\ & + [(\underline{R} - \underline{P}) \cdot \underline{u}_A]\underline{u}_A + \underline{u}_A S_{Aj} + \underline{P} \end{aligned} \quad (5)$$

$$\begin{aligned} \underline{R}'_j = & \cos \theta_{Bj} [(\underline{R} - \underline{Q}) - ((\underline{R} - \underline{Q}) \cdot \underline{u}_B)\underline{u}_B] + \sin \theta_{Bj} (\underline{u}_B \times (\underline{R} - \underline{Q})) \\ & + [(\underline{R} - \underline{Q}) \cdot \underline{u}_B]\underline{u}_B + \underline{u}_B S_{Bj} + \underline{Q} \end{aligned} \quad (6)$$

Using the identity $[\underline{A} - (\underline{A} \cdot \underline{u}_A)\underline{u}_A] = (\underline{u}_A \times \underline{A}) \times \underline{u}_A$, introducing the vectors:

$$\begin{aligned} \underline{K} &= \underline{R} - \underline{P} \\ \underline{L} &= \underline{R} - \underline{Q} \end{aligned} \quad (7)$$

and the following notation for any two vectors \underline{u}_C and \underline{D} ,

$$\underline{U}_{CD} = (\underline{u}_C \times \underline{D}) \times \underline{u}_C, \quad (7a)$$

we can substitute equations (7) and (7a) into equations (5) and (6) to get

$$\underline{R}_j = \underline{R} + \underline{u}_A S_A + (\cos\theta_{Aj} - 1)\underline{U}_{AK} + \sin\theta_{Aj}(\underline{u}_A \times \underline{K}) \quad (5a)$$

and

$$\underline{R}'_j = \underline{R} + \underline{u}_B S_B + (\cos\theta_{Bj} - 1)\underline{U}_{BL} + \sin\theta_{Bj}(\underline{u}_B \times \underline{L}) \quad (6a)$$

We now take the time-derivatives of equations for \underline{R}_j and \underline{R}'_j and using the notation of dots above the variables to indicate time derivatives, we obtain the following equations;

$$\dot{\underline{R}}_j = \underline{u}_A \dot{S}_{Aj} + [\cos\theta_{Aj}(\underline{u}_A \times \underline{K}) - \sin\theta_{Aj}\underline{U}_{AK}] \dot{\theta}_{Aj} \quad (8)$$

$$\dot{\underline{R}}'_j = \underline{u}_B \dot{S}_{Bj} + [\cos\theta_{Bj}(\underline{u}_B \times \underline{L}) - \sin\theta_{Bj}\underline{U}_{BL}] \dot{\theta}_{Bj} \quad (9)$$

$$\begin{aligned} \ddot{\underline{R}}_j = & \underline{u}_A \ddot{S}_{Aj} - [\cos\theta_{Aj} \underline{U}_{AK} + \sin\theta_{Aj}(\underline{u}_A \times \underline{K})] \dot{\theta}_{Aj}^2 \\ & + [\cos\theta_{Aj}(\underline{u}_A \times \underline{K}) - \sin\theta_{Aj} \underline{U}_{AK}] \ddot{\theta}_{Aj} \end{aligned} \quad (10)$$

$$\begin{aligned} \ddot{\underline{R}}'_j = & \underline{u}_B \ddot{S}_{Bj} - [\cos\theta_{Bj} \underline{U}_{BL} + \sin\theta_{Bj}(\underline{u}_B \times \underline{L})] \dot{\theta}_{Bj}^2 \\ & + [\cos\theta_{Bj}(\underline{u}_B \times \underline{L}) - \sin\theta_{Bj} \underline{U}_{BL}] \ddot{\theta}_{Bj} \end{aligned} \quad (11)$$

Substituting equation (7a) into equation (4), using equations (5a) and (6a), and by making the following substitutions:

$$\begin{aligned} \underline{M}_{Aj} = & \cos\theta_{Aj}(\underline{u}_A \times \underline{K}) - \sin\theta_{Aj}\underline{U}_{AK} \\ \underline{M}_{Bj} = & \cos\theta_{Bj}(\underline{u}_B \times \underline{L}) - \sin\theta_{Bj}\underline{U}_{BL} \end{aligned} \quad (12)$$

$$\underline{N}_{Aj} = \cos\theta_{Aj}\underline{U}_{AK} + \sin\theta_{Aj}(\underline{u}_A \times \underline{K}) \quad (13)$$

$$\underline{N}_{Bj} = \cos\theta_{Bj}\underline{U}_{BL} + \sin\theta_{Bj}(\underline{u}_B \times \underline{L})$$

we can derive the following *Working equations*:

$$\underline{\tilde{A}}'_j = \underline{\tilde{A}} + (\cos\theta_{Bj} - 1)\underline{\tilde{U}}_{BA} + \sin\theta_{Bj}(\underline{\tilde{u}}_B \times \underline{\tilde{A}}) \quad (14)$$

$$\dot{\underline{\tilde{A}}}'_j = [\cos\theta_{Bj}(\underline{\tilde{u}}_B \times \underline{\tilde{A}}) - \sin\theta_{Bj}\underline{\tilde{U}}_{BA}] \dot{\theta}_{Bj} = \underline{\tilde{V}}_{Bj} \dot{\theta}_{Bj} \quad (15)$$

$$\begin{aligned} \ddot{\underline{\tilde{A}}}'_j &= [\cos\theta_{Bj}(\underline{\tilde{u}}_B \times \underline{\tilde{A}}) - \sin\theta_{Bj}\underline{\tilde{U}}_{BA}] \ddot{\theta}_{Bj} \\ &\quad - [\cos\theta_{Bj}\underline{\tilde{U}}_{BA} + \sin\theta_{Bj}(\underline{\tilde{u}}_B \times \underline{\tilde{A}})] \dot{\theta}_{Bj}^2 \\ \ddot{\underline{\tilde{A}}}'_j &= \underline{\tilde{V}}_{Bj} \ddot{\theta}_{Bj} - \underline{\tilde{W}}_{Bj} \dot{\theta}_{Bj}^2 \end{aligned} \quad (16)$$

where

$$\underline{\tilde{V}}_{Bj} = \cos\theta_{Bj}(\underline{\tilde{u}}_B \times \underline{\tilde{A}}) - \sin\theta_{Bj}\underline{\tilde{U}}_{BA}$$

and

$$\underline{\tilde{W}}_{Bj} = \cos\theta_{Bj}\underline{\tilde{U}}_{BA} + \sin\theta_{Bj}(\underline{\tilde{u}}_B \times \underline{\tilde{A}})$$

$$\begin{aligned} \underline{\tilde{R}}_j - \underline{\tilde{R}}'_j &= \underline{\tilde{u}}_A \underline{\tilde{S}}_{Aj} + (\cos\theta_{Aj} - 1)\underline{\tilde{U}}_{AK} + \sin\theta_{Aj}(\underline{\tilde{u}}_A \times \underline{\tilde{K}}) \\ &\quad - \underline{\tilde{u}}_B \underline{\tilde{S}}_{Bj} - (\cos\theta_{Bj} - 1)\underline{\tilde{U}}_{BL} - \sin\theta_{Bj}(\underline{\tilde{u}}_B \times \underline{\tilde{L}}) \end{aligned} \quad (17)$$

$$\dot{\underline{\tilde{R}}}_j - \dot{\underline{\tilde{R}}}'_j = \underline{\tilde{u}}_A \dot{\underline{\tilde{S}}}_{Aj} + \underline{\tilde{M}}_{Aj} \dot{\theta}_{Aj} - \underline{\tilde{u}}_B \dot{\underline{\tilde{S}}}_{Bj} - \underline{\tilde{M}}_{Bj} \dot{\theta}_{Bj} \quad (18)$$

$$\begin{aligned} \ddot{\underline{\tilde{R}}}_j - \ddot{\underline{\tilde{R}}}'_j &= \underline{\tilde{u}}_A \ddot{\underline{\tilde{S}}}_{Aj} - \underline{\tilde{N}}_{Aj} \dot{\theta}_{Aj}^2 + \underline{\tilde{M}}_{Aj} \ddot{\theta}_{Aj} - \underline{\tilde{u}}_B \ddot{\underline{\tilde{S}}}_{Bj} \\ &\quad + \underline{\tilde{N}}_{Bj} \dot{\theta}_{Bj}^2 - \underline{\tilde{M}}_{Bj} \ddot{\theta}_{Bj} \end{aligned} \quad (19)$$

Displacement Analysis:

To analyze the displacements of a particular 3-link one-degree-of-freedom mechanism containing either the Sp or Sg pair, we need only to take working equation (17), apply the constraints of the particular grounded pairs and then substitute the results into the following pair geometry constraint equations for displacements.

For the Sp pair,

$$(\underline{\tilde{R}}_j - \underline{\tilde{R}}'_j) \cdot \underline{\tilde{A}}'_j = 0 \quad (20)$$

For the Sg pair.

$$(\underline{R}_j - \underline{R}'_j) = \underline{A}'_j S_{Gj} \quad (21)$$

Observe that equations (20) and (21) are equations (1) and (2) with $n = 0$.

The cylindrical pairs used in the derivation may be forced to work as prismatic (P) pairs by letting $\theta \equiv 0$ or may be forced to work as revolute (R) pairs by letting $S \equiv 0$.

The P-Sp-P Case:

For this mechanism, we use $\theta_A \equiv \theta_B \equiv 0$ and equations (14) and (17) are simplified to

$$\underline{R}_j - \underline{R}'_j = \underline{u}_A S_{Aj} - \underline{u}_B S_{Bj}$$

and

$$\underline{A}'_j = \underline{A}$$

Substituting in equation (20), we get

$$(\underline{u}_A S_{Aj} - \underline{u}_B S_{Bj}) \cdot \underline{A} = 0 \quad (22)$$

which simplifies to the input/output equation

$$S_{Bj} = \frac{\underline{u}_A \cdot \underline{A}}{\underline{u}_B \cdot \underline{A}} S_{Aj} \quad (23)$$

The R-Sp-P Case:

θ_A is the input; S_B is the output and $\theta_B \equiv S_A \equiv 0$. Equations (14) and (17) with $\theta_B \equiv S_A \equiv 0$ substituted in equation (20) provide,

$$(-\underline{u}_B S_{Bj} + (\cos\theta_{Aj} - 1)\underline{U}_{AK} + \sin\theta_{Aj}(\underline{u}_A \times \underline{K})) \cdot \underline{A} = 0$$

After simplification we obtain

$$S_{Bj} = \frac{[(\cos\theta_{Aj} - 1)\underline{U}_{AK} + \sin\theta_{Aj}(\underline{u}_A \times \underline{K})] \cdot \underline{A}}{\underline{u}_B \cdot \underline{A}} \quad (24)$$

The R-Sp-R Case

We have for this case $S_A \equiv S_B \equiv 0$, and equation (14) and (17) are simplified to obtain,

$$\begin{aligned} \underline{R}_j - \underline{R}'_j &= (\cos\theta_{Aj} - 1)\underline{U}_{AK} + \sin\theta_{Aj}(\underline{u}_A \times \underline{K}) - (\cos\theta_{Bj} - 1)\underline{U}_{BL} \\ &\quad - \sin\theta_{Bj}(\underline{u}_B \times \underline{L}) \end{aligned}$$

and

$$\underline{A}'_j = \underline{A} + (\cos\theta_{Bj} - 1)\underline{U}_{BA} + \sin\theta_{Bj}(\underline{u}_B \times \underline{A})$$

Substituting the above equations into equation (20); and simplifying the resulting equation, we obtain,

$$\begin{aligned} -\underline{S}_j \cdot \underline{A} + (\cos\theta_{Bj} - 1)[\underline{U}_{BL} \cdot \underline{A} - \underline{S}_j \cdot \underline{U}_{BA} - 2(\underline{u}_B \times \underline{L}) \cdot (\underline{u}_B \times \underline{A})] \\ + \sin\theta_{Bj}[(\underline{u}_B \times \underline{L}) \cdot \underline{A} - \underline{S}_j \cdot (\underline{u}_B \times \underline{A})] = 0 \end{aligned} \quad (25)$$

where \underline{S}_j is the known vector:

$$\underline{S}_j = (\cos\theta_{Aj} - 1)\underline{U}_{AK} + \sin\theta_{Aj}(\underline{u}_A \times \underline{K}) \quad (26)$$

Equation (25) can be solved for θ_{Bj} by using the following identities:

$$\cos\theta_B = \frac{1 - \tan^2 \frac{\theta_B}{2}}{1 + \tan^2 \frac{\theta_B}{2}}; \quad \sin\theta_B = \frac{2 \tan \frac{\theta_B}{2}}{1 + \tan^2 \frac{\theta_B}{2}} \quad (27)$$

and simplifying the resulting quadratic equation to yield:

$$\tan \frac{\theta_B}{2} j = \frac{-b \pm \sqrt{b^2 - c(C - 2a)}}{c - 2a} \quad (28)$$

$$\text{where: } a = \underline{U}_{BL} \cdot \underline{A} - \underline{S}_j \cdot \underline{U}_{BA} - 2(\underline{u}_B \times \underline{L}) \cdot (\underline{u}_B \times \underline{A})$$

$$b = (\underline{u}_B \times \underline{L}) \cdot \underline{A} - \underline{S}_j \cdot (\underline{u}_B \times \underline{A})$$

$$c = -\underline{S}_j \cdot \underline{A}$$

The P-Sp-R Case

Here, $\theta_A \equiv S_B \equiv 0$ and we have

$$\underline{R}_j - \underline{R}'_j = \underline{u}_A S_{Aj} - (\cos\theta_{Bj} - 1)\underline{U}_{BL} - \sin\theta_{Bj}(\underline{u}_B \times \underline{L})$$

and

$$\underline{A}'_j = \underline{A} + (\cos\theta_{Bj} - 1)\underline{U}_{BA} + \sin\theta_{Bj}(\underline{u}_B \times \underline{L})$$

Substituting the equations above into equation (20) and simplifying, we get,

$$\begin{aligned} & (\cos\theta_{Bj} - 1)[\underline{U}_{BL} \cdot \underline{A} - S_{Aj}(\underline{u}_A \cdot \underline{U}_{BA}) - 2(\underline{u}_B \times \underline{L}) \cdot (\underline{u}_B \times \underline{A})] \\ & + \sin\theta_{Bj}[(\underline{u}_B \times \underline{L}) \cdot \underline{A} - S_{Aj}\underline{u}_A \cdot (\underline{u}_B \times \underline{A})] - S_{Aj}\underline{u}_A \cdot \underline{A} = 0 \end{aligned} \quad (29)$$

Substituting equations (27) in equation (29) and simplifying the resulting quadratic gives us:

$$\tan \frac{\theta_{Bj}}{2} = \frac{-b \pm \sqrt{b^2 - c(c - 2a)}}{c - 2a} \quad (30)$$

where this time:

$$a = -\underline{U}_{BL} \cdot \underline{A} - S_{Aj}(\underline{u}_A \cdot \underline{U}_{BA}) - 2(\underline{u}_B \times \underline{L}) \cdot (\underline{u}_B \times \underline{A})$$

$$b = (\underline{u}_B \times \underline{L}) \cdot \underline{A} - S_{Aj}\underline{u}_A \cdot (\underline{u}_B \times \underline{A})$$

$$c = -S_{Aj}\underline{u}_A \cdot \underline{A}$$

The R-Sg-C Case

Only S_A in equation (17) is identically zero, so we get

$$\underline{R}_j - \underline{R}'_j = -\underline{u}_B S_{Bj} + \underline{S}_j - (\cos\theta_{Bj} - 1)\underline{U}_{BL} - \sin\theta_{BL}(\underline{u}_B \times \underline{L})$$

where \underline{S}_j is given by equation (26). Also,

$$\underline{A}'_j = \underline{A} + (\cos\theta_{Bj} - 1)\underline{U}_{BL} + \sin\theta_{Bj}(\underline{u}_B \times \underline{A})$$

Substituting in equation (21), we have

$$\underline{u}_B S_{Bj} - \underline{S}_j + (\cos\theta_{Bj} - 1)\underline{U}_{BL} + \sin\theta_{Bj}(\underline{u}_B \times \underline{L}) + \underline{A}'_j S_{Gj} = 0 \quad (31)$$

Taking the dot product of equation (31) with $\underline{A}'_j \times \underline{u}_B$ and upon sim-

plification, we get

$$\begin{aligned} & \cos\theta_{Bj} [S_j \cdot (\underline{A} \times \underline{u}_B) + U_{BL} \cdot (\underline{A} \times \underline{u}_B)] + \sin\theta_{Bj} [S_j \cdot \underline{U}_{BA} + \\ & U_{BL} \cdot \underline{U}_{BA}] - (\underline{u}_B \times \underline{L}) \cdot \underline{U}_{BA} = 0 \end{aligned} \quad (32)$$

Again, θ_{Bj} can be obtained by substituting equations (27) into equation (32) to obtain a quadratic whose solutions are:

$$\tan \frac{\theta_{Bj}}{2} = \frac{-b \pm \sqrt{a^2 + b^2 - c^2}}{c - a} \quad (33)$$

$$\text{where } a = S_j \cdot (\underline{A} \times \underline{u}_B) + U_{BL} \cdot (\underline{A} \times \underline{u}_B)$$

$$b = S_j \cdot \underline{U}_{BA} + U_{BL} \cdot \underline{U}_{BA}$$

$$c = -(\underline{u}_B \times \underline{L}) \cdot \underline{U}_{BA}$$

Taking the dot product of equation (31) with $(\underline{u}_B \times \underline{L})$ and simplifying, we get

$$\begin{aligned} S_{Gj} &= \frac{[S_j \cdot (\underline{u}_B \times \underline{L}) - (\cos\theta_{Bj} - 1)U_{BL} \cdot (\underline{u}_B \times \underline{L})]}{A'_j \cdot (\underline{u}_B \times \underline{L})} \\ &- \frac{\sin\theta_{Bj}(\underline{u}_B \times \underline{L}) \cdot (\underline{u}_B \times \underline{L})}{A'_j \cdot (\underline{u}_B \times \underline{L})} \end{aligned} \quad (34)$$

Taking the dot product of equation (31) with \underline{u}_B and simplifying, we get

$$S_{Bj} = [S_j - (\cos\theta_{Bj} - 1)U_{BL} - \sin\theta_{Bj}(\underline{u}_B \times \underline{L}) - S_{Gj}A'_j] \cdot \underline{u}_B \quad (35)$$

Velocity and Acceleration Analysis

To obtain the velocity and acceleration relations, we can either a), take the derivatives with respect to time of the displacement equations or b), use the higher order constraint equations. For the P-Sp-P case, taking the derivative of the displacement equation is trivial. But for the other cases, this procedure is cumbersome. It is therefore more convenient to just use equations (14) to (19)

in the following constraint equations (36) through (39), which are equations (1) and (2) with $n = 1$ and $n = 2$.

For the Sp pair

$$(\dot{\tilde{R}}_j - \dot{\tilde{R}}'_j) \cdot \tilde{A}'_j + (\tilde{R}_j - \tilde{R}'_j) \cdot \dot{\tilde{A}}'_j = 0 \quad (36)$$

and

$$(\ddot{\tilde{R}}_j - \ddot{\tilde{R}}'_j) \cdot \tilde{A}'_j + 2(\dot{\tilde{R}}_j - \dot{\tilde{R}}'_j) \cdot \dot{\tilde{A}}'_j + (\tilde{R}_j - \tilde{R}'_j) \cdot \ddot{\tilde{A}}'_j = 0 \quad (37)$$

For the Sg pair,

$$\dot{\tilde{R}}_j - \dot{\tilde{R}}'_j = \tilde{A}'_j \dot{\tilde{S}}_{Gj} + \dot{\tilde{A}}'_j S_{Gj} \quad (38)$$

and

$$\ddot{\tilde{R}}_j - \ddot{\tilde{R}}'_j = \ddot{\tilde{A}}'_j S_{Gj} + 2\dot{\tilde{A}}'_j \dot{\tilde{S}}_{Gj} + \tilde{A}'_j \ddot{S}_{Gj} \quad (39)$$

The P-Sp-P Case

Here we can use the time derivatives of the displacement equation to get:

$$S_{Bj} = \frac{\tilde{u}_A \cdot \tilde{A}}{\tilde{u}_B \cdot \tilde{A}} S_{Aj}$$

$$\dot{S}_{Bj} = \frac{\tilde{u}_A \cdot \dot{\tilde{A}}}{\tilde{u}_B \cdot \dot{\tilde{A}}} \dot{S}_{Aj} \quad (40)$$

$$\ddot{S}_{Bj} = \frac{\tilde{u}_A \cdot \ddot{\tilde{A}}}{\tilde{u}_B \cdot \ddot{\tilde{A}}} \ddot{S}_{Aj} \quad (41)$$

The R-Sp-P Case

Equations (18) and (19) become:

$$\dot{\tilde{R}}_j - \dot{\tilde{R}}'_j = \tilde{M}_{Aj} \dot{\tilde{\theta}}_{Aj} - \tilde{u}_B \dot{\tilde{S}}_{Bj}$$

and

$$\ddot{\tilde{R}}_j - \ddot{\tilde{R}}'_j = -\tilde{N}_{Aj} \dot{\tilde{\theta}}_{Aj}^2 + \tilde{M}_{Aj} \ddot{\tilde{\theta}}_{Aj} - \tilde{u}_B \ddot{\tilde{S}}_{Bj}$$

also

$$\tilde{A}'_j = \tilde{A}_j, \quad \dot{\tilde{A}}'_j = \dot{\tilde{A}}_j = 0$$

Substituting in equations (36) and (37), we get

$$\dot{\tilde{S}}_{Bj} = \frac{\tilde{M}_{Aj} \cdot \tilde{A}}{\tilde{u}_B \cdot \tilde{A}} \dot{\tilde{\theta}}_{Aj} \quad (42)$$

and

$$\ddot{\tilde{S}}_{Bj} = \frac{\tilde{N}_{Aj} \cdot \tilde{A}}{\tilde{u} \cdot \tilde{A}} \dot{\tilde{\theta}}_{Aj}^2 + \frac{\tilde{M}_{Aj} \cdot \tilde{A}}{\tilde{u}_B \cdot \tilde{A}} \ddot{\tilde{\theta}}_{Aj} \quad (43)$$

The R-Sp-R Case

$$S_{Aj} \equiv S_{Bj} \equiv \dot{S}_{Aj} \equiv \dot{S}_{Bj} \equiv \ddot{S}_{Aj} \equiv \ddot{S}_{Bj} \equiv 0$$

Equations (18) and (19) become:

$$\dot{\tilde{R}}_j - \dot{\tilde{R}}'_j = \tilde{M}_{Aj} \dot{\tilde{\theta}}_{Aj} - \tilde{M}_{Bj} \dot{\tilde{\theta}}_{Bj}$$

and

$$\ddot{\tilde{R}}_j - \ddot{\tilde{R}}'_j = -\tilde{N}_{Aj} \dot{\tilde{\theta}}_{Aj}^2 + \tilde{M}_{Aj} \ddot{\tilde{\theta}}_{Aj} + \tilde{N}_{Bj} \dot{\tilde{\theta}}_{Bj}^2 - \tilde{M}_{Bj} \ddot{\tilde{\theta}}_{Bj}$$

Also,

$$\tilde{A}'_j = \tilde{v}_{Bj} \dot{\tilde{\theta}}_{Bj} \quad ; \quad \dot{\tilde{A}}'_j = \tilde{v}_{Bj} \ddot{\tilde{\theta}}_{Bj} - \tilde{w}_{Bj} \dot{\tilde{\theta}}_{Bj}^2$$

Substituting in equations (36) and (37), we get

$$\dot{\tilde{\theta}}_{Bj} = \frac{\tilde{M}_{Aj} \cdot \tilde{A}'_j}{\tilde{M}_{Bj} \cdot \tilde{A}'_j - (\tilde{R}_j - \tilde{R}'_j) \cdot \tilde{v}_{Bj}} \dot{\tilde{\theta}}_{Aj} \quad (44)$$

and

$$\ddot{\theta}_{Bj} = \frac{-\tilde{N}_{Aj} \cdot \tilde{A}'_j}{D} \dot{\theta}_{Aj}^2 + \frac{\tilde{M}_{Aj} \cdot \tilde{A}'_j}{D} \ddot{\theta}_{Aj} + \frac{2\tilde{M}_{Aj} \cdot \tilde{V}_{Bj}}{D} \dot{\theta}_{Aj} \dot{\theta}_{Bj} + \frac{\tilde{N}_{Bj} \cdot \tilde{A}'_j - 2\tilde{M}_{Bj} \cdot \tilde{V}_{Bj} - (\tilde{R}_j - \tilde{R}'_j) \cdot \tilde{W}_{Bj}}{D} \dot{\theta}_{Bj}^2 \quad (45)$$

$$\text{where } D = \tilde{M}_{Aj} \cdot \tilde{A}'_j - (\tilde{R}_j - \tilde{R}'_j) \cdot \tilde{V}_{Bj} \quad (46)$$

The P-Sp-R Case

Equations (18) and (19) are

$$\tilde{R}_j - \tilde{R}'_j = \tilde{u}_A \dot{S}_{Aj} - \tilde{M}_{Bj} \dot{\theta}_{Bj}$$

$$\ddot{R}_j - \ddot{R}'_j = \tilde{u}_A \ddot{S}_{Aj} + \tilde{N}_{Bj} \dot{\theta}_{Bj}^2 - \tilde{M}_{Bj} \ddot{\theta}_{Bj}$$

also,

$$\dot{A}'_j = \tilde{V}_{Bj} \dot{\theta}_{Bj} \quad \text{and} \quad \ddot{A}'_j = \tilde{V}_{Bj} \ddot{\theta}_{Bj} - \tilde{W}_{Bj} \dot{\theta}_{Bj}^2$$

Substituting in equations (36) and (37) we get

$$\dot{\theta}_{Bj} = \frac{\tilde{u}_A \cdot \tilde{A}'_j}{\tilde{M}_{Bj} \cdot \tilde{A}'_j - (\tilde{R}_j - \tilde{R}'_j) \cdot \tilde{V}_{Bj}} \dot{S}_{Aj} \quad (47)$$

and

$$\ddot{\theta}_{Bj} = \frac{1}{D} [\tilde{u}_A \ddot{S}_{Aj} + 2(\tilde{u}_A \cdot \tilde{V}_{Bj}) \dot{S}_{Aj} \dot{\theta}_{Bj} + (\tilde{N}_{Bj} \cdot \tilde{A}'_j - 2\tilde{M}_{Bj} \cdot \tilde{V}_{Bj} - (\tilde{R}_j - \tilde{R}'_j) \cdot \tilde{W}_{Bj}) \dot{\theta}_{Bj}^2] \quad (48)$$

where D is given by equation (46).

The R-Sg-C Case

Only S_{Aj} , \dot{S}_{Aj} and \ddot{S}_{Aj} are zero and equations (18) and (19) become:

$$\tilde{R}_j - \tilde{R}'_j = \tilde{M}_{Aj} \dot{\theta}_{Aj} - \tilde{u}_B \dot{S}_{Bj} - \tilde{M}_{Bj} \dot{\theta}_{Bj}$$

and

$$\ddot{\tilde{R}}_j - \ddot{\tilde{R}}'_j = -\tilde{N}_{Aj} \dot{\theta}_{Aj}^2 + \tilde{M}_{Aj} \ddot{\theta}_{Aj} - \tilde{u}_B \ddot{S}_{Bj} + \tilde{N}_{Bj} \dot{\theta}_{Bj}^2 - \tilde{M}_{Bj} \ddot{\theta}_{Bj}$$

also,

$$\dot{\tilde{A}}'_j = \tilde{V}_{Bj} \dot{\theta}_{Bj} \quad \text{and} \quad \ddot{\tilde{A}}'_j = \tilde{V}_{Bj} \ddot{\theta}_{Bj} - \tilde{W}_{Bj} \dot{\theta}_{Bj}^2$$

Substituting the expression for $(\dot{\tilde{R}}_j - \dot{\tilde{R}}'_j)$ just obtained into equation (38) we get

$$\tilde{M}_{Aj} \dot{\theta}_{Aj} - \tilde{u}_B \dot{S}_{Bj} - \tilde{M}_{Bj} \dot{\theta}_{Bj} = \tilde{A}'_j \dot{S}_{Gj} + \tilde{V}_{Bj} \dot{\theta}_{Bj} S_{Gj} \quad (49)$$

$\dot{\theta}_{Bj}$, \dot{S}_{Bj} and \dot{S}_{Gj} are unknowns in equation (49).

Taking the dot product of equation (49) with $(\tilde{A}'_j \times \tilde{u}_B)$, we get

$$(\tilde{M}_{Aj} \dot{\theta}_{Aj} - \tilde{M}_{Bj} \dot{\theta}_{Bj}) \cdot (\tilde{A}'_j \times \tilde{u}_B) = \tilde{V}_{Bj} \cdot (\tilde{A}'_j \times \tilde{u}_B) \dot{\theta}_{Bj} S_{Gj}$$

or

$$\dot{\theta}_{Bj} = \frac{\tilde{M}_{Aj} \cdot \tilde{A}'_j \times \tilde{u}_B}{(S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj}) \cdot \tilde{A}'_j \times \tilde{u}_B} \dot{\theta}_{Aj} \quad (50)$$

Now, taking the dot product of equation (49) with $(\tilde{A}'_j \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj}))$, we have,

$$(\tilde{M}_{Aj} \dot{\theta}_{Aj} - \tilde{u}_B \dot{S}_{Bj}) \cdot \tilde{A}'_j \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj}) = 0$$

$$\text{or} \quad \dot{S}_{Bj} = \frac{\tilde{M}_{Aj} \cdot \tilde{A}'_j \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj})}{\tilde{u}_B \cdot \tilde{A}'_j \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj})} \dot{\theta}_{Aj} \quad (51)$$

Again taking the dot product of equation (49) with

$$\tilde{u}_B \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj}),$$

we have $(\tilde{M}_{Aj} \dot{\theta}_{Aj} - \tilde{A}'_j \dot{S}_{Gj}) \cdot \tilde{u}_B \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj}) = 0$

$$\text{or} \quad \dot{S}_{Gj} = \frac{\tilde{M}_{Aj} \cdot \tilde{u}_B \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj})}{\tilde{A}'_j \cdot \tilde{u}_B \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj})} \dot{\theta}_{Aj} \quad (52)$$

Acceleration: Substituting the expression for $(\tilde{R}_j - \tilde{R}'_j)$ obtained earlier for the R-Sg-C case into equation (39), we will get,

$$-\tilde{N}_{Aj} \dot{\theta}_{Aj}^2 + \tilde{M}_{Aj} \ddot{\theta}_{Aj} - \tilde{u}_B \ddot{S}_{Bj} + \tilde{N}_{Bj} \dot{\theta}_{Bj}^2 - \tilde{M}_{Bj} \ddot{\theta}_{Bj} =$$

$$(\tilde{V}_{Bj} \ddot{\theta}_{Bj} - \tilde{W}_{Bj} \dot{\theta}_{Bj}^2) S_{Gj} + 2\tilde{V}_{Bj} \dot{\theta}_{Bj} \dot{S}_{Gj} + \tilde{A}'_j \ddot{S}_{Gj}$$

or

$$\tilde{u}_B \ddot{S}_{Bj} + \tilde{A}'_j \ddot{S}_{Gj} + (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj}) \ddot{\theta}_{Bj} = -\tilde{N}_{Aj} \dot{\theta}_{Aj}^2 + \tilde{M}_{Aj} \ddot{\theta}_{Aj} +$$

$$(\tilde{N}_{Bj} + S_{Gj} \tilde{W}_{Bj}) \dot{\theta}_{Bj}^2 - 2\tilde{V}_{Bj} \dot{\theta}_{Bj} \dot{S}_{Gj} \quad (53)$$

Letting \tilde{x} be equal to the right hand side of equation (53) and by using the same technique of taking the dot product of equation (53) with the proper cross-products, we will obtain the following,

$$\ddot{\theta}_{Bj} = \frac{\tilde{x} \cdot \tilde{A}'_j \times \tilde{u}_B}{(S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj}) \cdot \tilde{A}'_j \times \tilde{u}_B} \quad (54)$$

$$\ddot{S}_{Bj} = \frac{\tilde{x} \cdot \tilde{A}'_j \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj})}{\tilde{u}_B \cdot \tilde{A}'_j \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj})} \quad (55)$$

$$\ddot{S}_{Gj} = \frac{\tilde{x} \cdot \tilde{u}_B \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj})}{\tilde{A}'_j \cdot \tilde{u}_B \times (S_{Gj} \tilde{V}_{Bj} + \tilde{M}_{Bj})} \quad (56)$$

Numerical Examples

1. Displacement analysis of a R-Sp-R mechanism.

The vectors describing the mechanism are,

$$\tilde{u}_A = (2\hat{i} + 1\hat{j} + 0\hat{k}) \ 1 / \sqrt{5}$$

$$\tilde{u}_B = (-3\hat{i} + 2\hat{j} + 0\hat{k}) \ 1 / \sqrt{13}$$

$$\tilde{P} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\tilde{Q} = 0\hat{i} + 0\hat{j} + 1\hat{k}$$

$$\tilde{R} = 1.5\hat{i} + 1.5\hat{j} + 1.5\hat{k}$$

$$\tilde{A} = 0\hat{i} + 0\hat{j} + 1\hat{k}$$

Displacement analysis results

θ_A (degrees)	θ_B (degrees)
40	1.32
80	-8.48
120	-22.88
160	-36.30
200	-45.41
240	-48.06
280	-39.90
320	-17.82
355	-1.29

2. Displacement, velocity and acceleration analysis of a R-Sg-C mechanism.

The mechanism parameters are;

$$\underline{u}_B = (1\hat{i} + 2\hat{j} + 1\hat{k}) 1/\sqrt{6}$$

$$\underline{u}_B = (1\hat{i} + 1\hat{j} + 0\hat{k}) 1/\sqrt{2}$$

$$\underline{p} = 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$\underline{q} = 0\hat{i} + 0\hat{j} + 1\hat{k}$$

$$\underline{r} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\underline{a} = (1\hat{i} + 1\hat{j} + 2\hat{k}) 1/\sqrt{6}$$

The motion parameters are: $\dot{\theta}_{Aj}$ is one unit of angular velocity and θ_{Aj} is zero, both constant for $j = 0, 1, 2, \dots$

The results of the analysis for the R-Sg-C mechanism are shown in a table on the next page.

The direction of the rotations and linear motions are established using the right hand rule. Rotations are positive counter-clockwise looking at the head of the unit vectors \underline{U}_A and \underline{U}_B . Linear motions are positive when they are in the direction of the vectors they are associated with.

It is to be mentioned here also that although the quadratic equations gave two sets of solutions, only one set will define the motion of the mechanism. The other set of solutions are for those positions in which the mechanism has to be disassembled into the other possible configuration.

Conclusions

Displacements, velocities and accelerations have been derived for several three-link spatial mechanisms containing sphere-plane and sphere-groove pairs. The groove of the sphere-groove pair was assumed to be a cylindrical groove, resulting in straight line axis of the groove. However, a more generalized groove may be one whose axis is a spatial curve. The authors are working on developing analysis procedures for mechanisms containing such a generalized sphere-groove pair. The expected results of their work will be subject of a forthcoming paper. Similarly, the authors also have the generalization of the sphere-plane pair in progress, in which the parallel planes of the pair are generalized to form equidistant curve surfaces.

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Table of displacement, velocities and accelerations.

θ_A	θ_B	$\dot{\theta}_B$	$\ddot{\theta}_B$	S_G	\dot{S}_G	\ddot{S}_G	S_B	\dot{S}_B	\ddot{S}_B
40	12.99	.97	.87	-1.22	-1.56	.73	1.26	1.57	-.98
80	-95.87	-3.23	2.66	-2.06	-.06	5.28	2.04	.18	-3.90
120	-112.24	.05	.96	-.64	.53	-.67	1.11	-.74	-.36
160	-98.34	.40	.27	.58	.83	-1.02	-.04	-1.29	.29
200	-81.04	.42	-.03	1.32	1.26	-1.21	-1.06	-1.54	.99
240	-61.47	.29	-.12	1.55	1.24	-1.56	-1.65	-1.15	1.65
280	-40.35	.16	-.17	1.37	.43	-1.67	-1.65	-.10	1.82
320	-19.17	.18	-.29	.86	-.74	-1.48	-1.05	1.09	1.41
355	-2.2	.39	-.22	.13	-1.45	-.90	-.15	1.7	.59

Appendix

1. Sphere-plane Constraint Equation

The complete displacement constraint equations of the Sp-pair are:

$$\tilde{R}_j - \tilde{R}'_j = S_{Pj} \tilde{u}_{Pj} \quad (a)$$

and

$$\tilde{u}'_{Pj} \cdot \tilde{A}'_{Pj} = 0 \quad (b)$$

where \underline{u}'_{Pj} is a unit vector in the plane of the Sp pair, perpendicular to \underline{A}'_j and is in the direction of the relative motion of point R of body 1 with respect to the initially coincident point R' of body 2.

Derivatives of equations a) and b) with respect to time are taken to give the following velocity and acceleration constraint equations:

$$\text{Velocity: } \dot{\underline{R}}_j - \dot{\underline{R}}'_j = \dot{S}_{Pj} \underline{u}'_{Pj} + S_{Pj} \dot{\underline{u}}'_{Pj} \quad (c)$$

$$\text{and } \dot{\underline{u}}'_{Pj} \cdot \underline{A}'_j + \underline{u}'_{Pj} \cdot \dot{\underline{A}}_j = 0 \quad (d)$$

$$\text{Acceleration: } \ddot{\underline{R}}_j - \ddot{\underline{R}}'_j = \ddot{S}_{Pj} \underline{u}'_{Pj} - 2\dot{S}_{Pj} \dot{\underline{u}}'_{Pj} + S_{Pj} \ddot{\underline{u}}'_{Pj} \quad (e)$$

$$\text{and } \ddot{\underline{u}}'_{Pj} \cdot \underline{A}'_j + 2\dot{\underline{u}}'_{Pj} \cdot \dot{\underline{A}}_j + \underline{u}'_{Pj} \cdot \ddot{\underline{A}}_j = 0 \quad (f)$$

The constraint equations a) to f) are complete in the sense that all of the important variables in the motion of the joint elements are included. Also, the Coriolis component in the acceleration constraint equation f) is evident since $\dot{\underline{A}}_j$ is a function of $\dot{\theta}_{Bj}$.

2. Proof that $\frac{d^n}{dt^n} (\underline{R}_j - \underline{R}'_j) \cdot \underline{A}'_j = 0$ $n = 0, 1, 2$ satisfies

the complete Sp pair constraints equation.

Without loss of generality, we can let $\underline{S}_{Pj} = S_{Pj} \underline{u}'_{Pj}$ and write the complete constraint equation as:

$$\frac{d^n}{dt^n} (\underline{R}_j - \underline{R}'_j) = \frac{d^n}{dt^n} (\underline{S}_{Pj}) \quad a)$$

and

$$\frac{d^n}{dt^n} (\underline{S}_{Pj} \cdot \underline{A}'_j) = 0 \quad b)$$

Displacement: For $n = 0$, equation a) and b) are

$$(\underline{R}_j - \underline{R}'_j) = \underline{S}_{Pj} \quad c)$$

and

$$\underline{\underline{S}}_{Pj} \cdot \underline{\underline{A}}'_j = 0 \quad \text{d)}$$

Taking the dot product of equation c) with $\underline{\underline{A}}'_j$ gives us the displacement constraint equation for the Sp pair.

$$(\underline{\underline{R}}_j - \underline{\underline{R}}'_j) \cdot \underline{\underline{A}}'_j = 0 \quad \text{e)}$$

Velocity: With $n = 1$, equations a) and b) will become

$$\underline{\underline{\dot{R}}}_j - \underline{\underline{\dot{R}}}'_j = \underline{\underline{\dot{S}}}_{Pj} \quad \text{f)}$$

and

$$\underline{\underline{\dot{S}}}_{Pj} \cdot \underline{\underline{A}}'_j = -\underline{\underline{S}}_{Pj} \cdot \underline{\underline{\dot{A}}}'_j \quad \text{g)}$$

Taking the dot product of equation f) with $\underline{\underline{A}}'_j$ gives us

$$(\underline{\underline{\dot{R}}}_j - \underline{\underline{\dot{R}}}'_j) \cdot \underline{\underline{A}}'_j = \underline{\underline{\dot{S}}}_{Pj} \cdot \underline{\underline{A}}'_j \quad \text{h)}$$

Substituting equation g) into h), we will have

$$(\underline{\underline{\dot{R}}}_j - \underline{\underline{\dot{R}}}'_j) \cdot \underline{\underline{A}}'_j = -\underline{\underline{S}}_{Pj} \cdot \underline{\underline{\dot{A}}}'_j \quad \text{i)}$$

Equation c) can now be substituted in equation i) to get:

$$(\underline{\underline{\dot{R}}}_j - \underline{\underline{\dot{R}}}'_j) \cdot \underline{\underline{A}}'_j = -(\underline{\underline{R}}_j - \underline{\underline{R}}'_j) \cdot \underline{\underline{\dot{A}}}'_j$$

or

$$(\underline{\underline{\dot{R}}}_j - \underline{\underline{\dot{R}}}'_j) \cdot \underline{\underline{A}}'_j + (\underline{\underline{R}}_j - \underline{\underline{R}}'_j) \cdot \underline{\underline{\dot{A}}}'_j = 0 \quad \text{j)}$$

$$\text{which is really } \frac{d}{dt} [(\underline{\underline{R}}_j - \underline{\underline{R}}'_j) \cdot \underline{\underline{A}}'_j] = 0 \quad \text{k)}$$

Acceleration: For $n = 2$, equations a) and b) will be

$$\underline{\underline{\ddot{R}}}_j - \underline{\underline{\ddot{R}}}'_j = \underline{\underline{\ddot{S}}}_{Pj} \quad \text{l)}$$

and

$$\underline{\underline{\ddot{S}}}_{Pj} \cdot \underline{\underline{A}}'_j + 2\underline{\underline{\dot{S}}}_{Pj} \cdot \underline{\underline{\dot{A}}}'_j + \underline{\underline{S}}_{Pj} \cdot \underline{\underline{\ddot{A}}}'_j = 0 \quad \text{m)}$$

Taking the dot product of equation l) with $\underline{\underline{A}}'_j$ and substituting

$$\ddot{\tilde{S}}_{Pj} \cdot \tilde{A}'_j = -2\dot{\tilde{S}}_{Pj} \cdot \dot{\tilde{A}}'_j - \tilde{S}_{Pj} \cdot \ddot{\tilde{A}}'_j$$

from equation m), we will get

$$(\ddot{\tilde{R}}_j - \ddot{\tilde{R}}'_j) \cdot \tilde{A}'_j = -2\dot{\tilde{S}}_{Pj} \cdot \dot{\tilde{A}}'_j - \tilde{S}_{Pj} \cdot \ddot{\tilde{A}}'_j$$

$$\text{or } (\ddot{\tilde{R}}_j - \ddot{\tilde{R}}'_j) \cdot \tilde{A}'_j + 2\dot{\tilde{S}}_{Pj} \cdot \dot{\tilde{A}}'_j + \tilde{S}_{Pj} \cdot \ddot{\tilde{A}}'_j = 0 \quad \text{n)}$$

Substituting equations c) and f) into equation n) gives us,

$$(\ddot{\tilde{R}}_j - \ddot{\tilde{R}}'_j) \cdot \dot{\tilde{A}}'_j + 2(\dot{\tilde{R}}_j - \dot{\tilde{R}}'_j) \cdot \dot{\tilde{A}}'_j + (\tilde{R}_j - \tilde{R}'_j) \cdot \ddot{\tilde{A}}'_j = 0$$

which is

$$\frac{d^2}{dt^2} I(\tilde{R}_j - \tilde{R}'_j) \cdot \tilde{A}'_j = 0 \quad \text{o)}$$

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