STRUCTURAL VIBRATION THEORY AND INTRODUCTION TO SEISMOLOGY

By

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The study of Earthquake Engineering is beat introduced by reference to a closely related science called Seismology which deals with the study of earthquakes and related phenomena. During the last 30 or so years, considerable knowledge has been collected regarding seismic activity — and these data have been interpreted by engineers in connection with the effect on structures. A proper understanding of this potentially powerful activity, therefore, is necessary if the interpretation has to be reasonable and accurate. It must be emphasized that Seismologists are indeed highly-trained scientists, and by training, their area of interest regarding earthquakes is quite different from that of the engineer. More particularly, a Seismologist may well record any small earthquake which may occur at any point on the earth; he may wish to learn more about the internal constitution of the earth and he is most interested in absolute times of travel of seismic waves. In his work, therefore he needs sensitive instruments of high magnification and most of the data he gathers may not be useful to the engineer. In contrast, the engineer is more interested in the ground motion triggered by earthquakes especially the type of ground response which causes damage to his structures. He will therefore need rugged devices which will record the largest shock near the vicinity of the instrument.

The work of the Seismologist has paved the way in identifying active earthquake areas all around the world. For example, a greater percentage of occurrences of earthquake take place along the fringes of the periphery of the Pacific Ocean — from the southernmost tip of western South America, all along the length of Chile, Peru, the central Americas, Mexico, western United States and Canada, Alaska—then coming down thru Japan, Okinawa, then the Philippines, northeast Australia to New Zealand. This is commonly called the Circum-Pacific belt. Another active region is the Alpide belt which runs east-west—from the Alps in Europe, thru Asia minor (Turkey, Iran) thru the Himalayas of northern India ending in Singapore, where it connects with the northern and southern orientation of the former belt. Minor Seismic areas include the Atlanta, Arctic and Indian Oceans. This is by no means saying that no earthquake may ever occur anywhere else!

The earthquake mechanism is now understood to be caused by the sudden release of pressures after a very slow process of building up strains within the earth's crust. When the strains within fault planes can no longer be restrained, displacements are the consequences, and in the process, Seismic waves are created which travel in all directions. There seems to be long periods of a so-called unstable equilibrium and the sudden release is often times triggered by many causes, among them atmospheric disturbances. Raid (19ññ) proposed the elastic rebound theory which essentially include the discussion described above. In any case, the mechanism originates at a relatively small volume of the earth's crust, called the FOCUS or HYPOCENTER of the earthquake, these points are frequently 10 to 50 km. deep although records have shown that there are a few earthquakes which originate at 600 km. The point at the surface of the earth directly above the focus is called the EPICENTER, after Richter (1958). The propagation stress waves is affected by the characteristics of known fault lines and it is important to emphasize that the engineer must be more concerned about the distance to this cansative fault rather than the distance to the epicenter. A shallow-focus earthquake is one whose focus is not more than about 50 km. deep.

The passage of seismic waves thru the earth's crust is a complex process. In order to have a workable knowledge of the propagation, seismic waves are oftentimes assumed to travel in an isotropic, hookean and homogeneous layers of soil. This is of course far from being accurate but a prediction based on these ideal assumptions is better than not having any guide at all. Scientists have started to take into account distortion of the wave fronts due to anisotropy of rocks and soil, their inelastic behavior and also the reflection and/or refraction of the wave which hits a boundary. And because of the presence of substructures which are already located in the ground, the accuracy of wave properties is not expected to be highly fixed. At any rate, seismic waves are classified into:

1. Body waves
   a. Dilatational, irrotational or P-waves are those whose oscillations are in the direction of the propagation of the wave front. They are characterized by volume changes in the earth and are the first to be recorded on seismographs because of their fast velocity, which is in the order of 20,000 feet/sec to 30,000 ft/sec.

b. Transverse, shear or S-waves oscillations in any plane normal to propagation. They are characterized by no volume changes in the earth but they possess rotational quality. They are slower than P-waves since their velocity is of the order of 10,000 feet/sec, but they transmit much more energy.

2. Surface waves are of minor importance but they are:
   a. Raleigh, or R-waves which travel at the surface of a solid and are combinations of the P- and S-waves described earlier above.
   b. Love, or Q-waves vibrate transversely but do not possess vertical components.

In 1935, Richter devised a Magnitude Scale in order to describe the size of an earthquake. The *magnitude* of an earthquake is the measure of the energy released by the shock. This is designated as M and is given by a number, which is defined as the logarithm (to the base 10) of the maximum trace amplitude in microns which a "standard" seismometer would record with an epicentral distance of 100 km. The "standard" instrument has a period of 0.8 seconds, a static magnification of 2,800 and is nearly at critical damping Stated mathematically:

$$M = \log_{10} \frac{A}{A_0}$$

where $A =$ max. trace amplitude and A_0 is the amplitude of one micron. The magnitude of an earthquake, therefore, is an instrumental measure and is a scientific physical quantity. Experience has shown that if $M = 5$ or greater, the ground motion generated are oftentimes severe to be potentially damaging to structures. In the absence of instrumental readings, severity of ground shaking is described by an *Intensity* scale. This assignment of an intensity number is not a precise engineering measure but nevertheless, a quantitative figure as well. The decision is based on three factors: the geologic conditions, the distance from the epicenter or from a causative fault, and the type of structure. A popular intensity scale is the Modified Mercalli proposed in 1931 and consisting of 12 degrees as follows:

I. Detected only by sensitive instruments;

II. Felt by few persons at rest, especially on upper floors; delicate suspended objects may swing.

III. Felt noticeably indoors, but not always recognized as a quake; standing autos rock slightly, vibration like passing truck.

IV. Felt indoors by many, outdoors by a few; at night some awaken; dishes, windows, doors disturbed; motor cars rock noticeably.

V. Felt by most people; some breakage of dishes, windows, and plaster; disturbance of tall objects.

VI. Felt by all; many are frightened and run outdoors; falling plaster and chimneys; damage small.

VII. Everybody runs outdoors; damage to buildings noticed by drivers of autos.

VIII. Panel walls thrown out of frames; fall of walls, monuments, chimneys; sand and mud ejected; drivers of autos disturbed.

IX. Buildings shifted off foundations, cracked, thrown out of plumb; ground cracked; underground pipes broken.

X. Most masonry and frame structures destroyed; ground cracked; rails bend; landslides.

XI. New structures remain standing; bridges destroyed; fissures in ground; pipes broken; landslides; rails bent.

XII. Damage total; waves seen on ground surface; lines of sight and level distorted; objects thrown up into air.

The Japanese earthquake-intensity scale prepared by the Central Meteorological Observatory (CMO) of Tokyo has endeavored to include an acceleration parameter in addition to the descriptions. The original version was divided into only VI degrees, but recently there is a plan to increase this to VII. The magnitude of ground acceleration assigned to degree VI is 512 gals, where 980 gals equals gravity acceleration g. The Rossi-Forel intensity scale consists of 10 degrees and they are based on the study of earthquakes observed in Italy.

Data on earthquakes in the Philippines is very scarce. For one thing, recording seismographs are very expensive instruments; for another, earthquakes do not happen as frequently as typhoons do—and even if they do occur, there is usually no warning at all, again, unlike typhoons. Instruments usually record displacements or accelerations but the latter type are preferred. It is important that at least two components are recorded for horizontal accelerations. One particular instrument will operate within a certain range of frequencies such that a record consists of a wave form which is proportional to ground acceleration. The important parameters of a record include: the duration of the earthquake, the magnitude, component direction, maximum ground acceleration, velocity and displacement. One of the most severe earthquakes on record happened in 1940 at El Centro, California, whereby the magnitude was 7.1, max. acceleration of the ground was 0.33 g, max. ground velocity = 13.7

inches/sec and max. ground displacement was 8.3 inches. The duration was well beyond 30 seconds with the most vigorous activity occurring for 12 seconds. It was estimated that the maximum relative displacement at or near the focus was of the order of 20 feet. The El Centro record has been used by many researchers in dynamic studies because of its "completeness". The probability that this earthquake magnitude will occur in a specific locality in California puts the occurrence at 50 years. The Taft (California) earthquake of 1966 was of magnitude 5 to 6, had a max. ground acceleration of 0.5 g but of much shorter duration, 12 seconds. As a matter of fact, the time of vigorous shaking was only 1-1/2 seconds. In general, max. ground accelerations decrease with distance from the causative fault. Vertical components of ground accelerations are about 1/3 to 2/3 of the horizontal components, but the former are about 50% higher frequency.

During the past decade, researchers have succeeded in generating data idealized "laboratory" earthquakes in the probabilistic sense. More recently, data recorded by many instruments during the San Fernando valley (Los Angeles, California) earthquake of Feb. 1971, are utilized to "replay" the earthquake to a reduced scale in the laboratory. These generation of simulated earthquake is a valuable tool in the dynamic analysis of models.

In the dynamic analysis of a structure, it is important that the structure be idealized as a mathematical model having a finite number of degrees of freedom. The lateral analysis is performed using the model by imposing statistically equivalent lateral loads without due regard to the influence of the foundation. In this Seminar an effort will be made to take important factors into account in a truly dynamic investigation. In any case, in the design of the structure to resist dynamic effects, the engineer must have the following in mind—

1. that the structure will survive without damage even a moderate earthquake.
2. that the structure will not suffer major damage as a result of the most severe earthquake predictable during the anticipated life of the structure, and
3. that the structure should not collapse even if it is subjected to one earthquake of abnormally strong intensity.

The theory of Structural Vibrations consists of writing the equations of motion of each particle of mass of the structure in order to study its vibratory characteristics. In technical language, the engineer wishes to determine the *response* of the structure due to some dynamic effects. Response may mean getting the displacements of the various masses at specific times, describing the relative movements of the masses, solving for the stress (or strain) variations due to the motions of the parts of the structure, or the like. Of ultimate interest to the engineer are the shears, axial forces and bending moments in the members—which quantities are the ones needed for design.

IDEALIZATION OF STRUCTURES INTO MATHEMATICAL MODELS; SINGLE-DEGREE-OF-FREEDOM SYSTEMS

One of the most important parameters in the study of vibrations is the so-called "degree-of-freedom". This is defined as the number of *independent* coordinates which are necessary to describe the configuration of the system. It is obvious that the description of every particle of mass consisting a structure will involve an infinite degrees of freedom. Mathematically, this can mean that an infinitesimal volume of mass may be analyzed—its equation of motion derived in the form of an equation. In order to draw up the contributions of all the other volumes, we need the integration of the equation. This method is far from being practical, and even assuming that the solution can be done at all, the procedure is cumbersome and time consuming. Furthermore, even if the problem is fed into a computer, some means of a numerical scheme must be devised in order to reduce the work into one which is feasible. This closed-solution utilizes a mathematical model which is described as a distributed-mass system, and the coordinates consistent with the assumption are called distributed coordinates. In view of the numerical difficulties, this method is only applied, for illustration, to the case of prismatic and straight beams.

A more convenient idealization is to assume that the mass of the structure are lumped at certain points only, and the description of the resulting response is made only for these lumped masses. The members are assumed to retain their stiffnesses (or flexibilities). The problem is thus reduced to one having a finite number of degrees of freedom. Of course it goes without saying that the more closely masses are lumped, the more accurate the solution — the limit of which decision is the distributed mass problem itself. To illustrate a distributed mass system, we take the case of a simply supported beam AB carrying a distributed load p(t,x), having a mass per unit length of the beam of m, and whose EI is constant. Such a beam is shown in Fig. 1. The free body diagram of an infinitesimal length dx of the beam is also shown. Noting that the y-axis is directed downward, differentiating twice with respect to time this assumed positive direction of displacement gives a positive direction of the acceleration also as downward. The jagged arrow therefore represents the reversed effective inertia force as propounded by D' Alembert. The equation of motion for the element is readily written along the y-direction:

$$my - \frac{dV}{dx} = p(t,x) \tag{1}$$

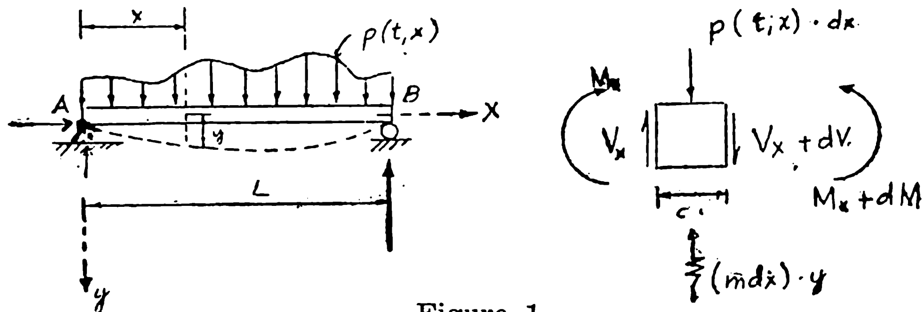


Figure 1

In strength of materials, we have learned that the curvature can be written in an approximate relation:

$$\frac{d^2y}{dx^2} = - \frac{M}{EI} \tag{2}$$

Differentiating Eq. (2) twice with respect to x;

$$- \frac{d2M}{dx^2} = \frac{dv}{dx} = EI \frac{d^4y}{dx^4} \tag{3}$$

We can new write the total differentials as partial differentials and substitute into Eq. (1) which yields

$$my + EI \frac{\partial^4 y}{\partial x^4} = p(t,x) \tag{4}$$

which is a fourth-order partial differential equation.

The simply supported beam may also be idealized as a lumped mass system shown in Fig. 2. The entire mass of the beam is lumped at a single point, say at the center, while the rest of the beam retains its flexibility. It is clear that a single coordinate its flexibility. It is clear that a single coordinate will describe completely the configuration of the system — and as such is a single-degree-of-freedom system, oftentimes abbreviated SDF. The girder, which includes the floor, of the single-bay bent shown in Fig. 3, has a mass which is very much greater compared to the masses of the two columns. The error will be small if we assume that all the mass is

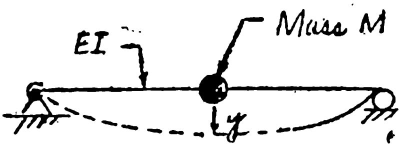


Figure 2

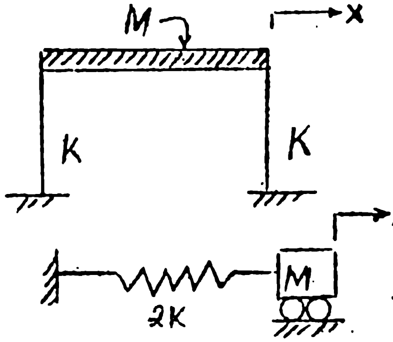


Fig. 3

concentrated at the girder level. A single coordinate x is necessary to define the vibration of the structure. Furthermore, if the girder is assumed very stiff in comparison to the stiffness of the column, there will be no rotation at the joints. The bent can also be represented by the spring mass system in Fig. 3. A tall tower may be idealized as a 5-degree of freedom system shown in Fig. 4 while a three-story frame is represented as a 3-degree-of freedom system in Fig. 5.

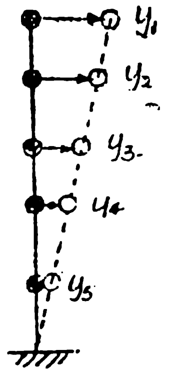
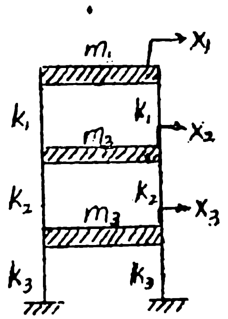


Fig. 4



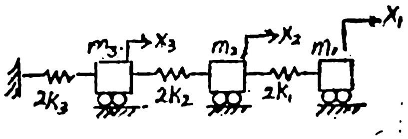


Fig. 5

The synthesis in the analysis of a multi-degree of freedom system (MDF-System) involves the superposition effects of as many SDF systems, and this makes the treatment of SDF systems an important one. The most general SDF idealization must include all effects due to structural considerations. Essentially, there must be a single lumped mass supported by a spring in a free vibration state. To take cognizance of the effect of damping, we introduced the concept of a damper, drawn as a dashpot representing a presence of a damping force which always opposes motion. The classical damping force is one which is classified as viscous—meaning to say, whose magnitude is proportional to velocity. This statement is written in formula form as

$$F_d = - c\dot{x} \tag{5}$$

where c is the damping constant. Lastly to complete the picture, a forcing function is introduced which is a function of time. This last inclusion classifies the problem as a *forced* vibration. The complete idealization of a SDF system is shown in Fig. 6 below. Drawing the freebody diagram of the mass and basing quantities on the

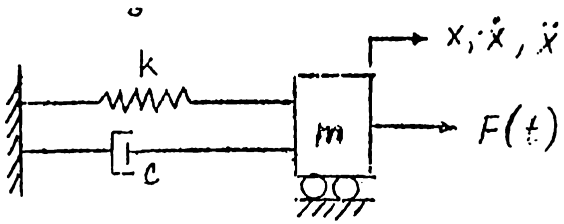


Fig. 6

assumed positive directions of x, $\dot{x}$ and $\ddot{x}$, the equation of motion of a *damped forced* vibration of a SDF system is which is a first degree,

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{6}$$

second order differential equation. If there is no damping, the term $c\dot{x}$ is deleted and the problem is *undamped.* If there is no forcing function, F(t)=0 and the vibration is considered *free.* Taking the simplest case of an undamped free SDF system, which is of course

$$m\ddot{x} + kx = 0 \tag{7}$$

a homogeneous differential equation. We now define a parameter $w^2 = \frac{k}{m}$ and write

$$\ddot{x} + w2x = 0 \tag{8}$$

whose solution is

$$x = A \cos wt + b \sin wt \tag{9}$$

Differentiating Eq. (9) twice with respect to time and substitute this value of x and Eq. (9) itself into Eq. (8), we can verify that Eq. (9) is indeed the solution.

It is instructive to determine the solution of the damped free case since this solution is also the complementary solution of the equation of the equation of Eq. (6). Writing

$$\ddot{x} + \frac{C}{m}\dot{x} + \frac{K}{m}x = 0 \tag{10}$$

we also define $\frac{c}{m} = 2\beta w$ and we have already defined $\frac{K}{m} = w2$. The new parameter β is called the damping coefficient which is the ratio $\frac{c}{Ccr}$. Critical damping which in itself is not too important to engineers particularly with respect to structures is defined as the constant which will damp out the vibration in the shortest possible time. Some engineers may argue that the critical damping case does not offer any vibrations at all. Of particular interest to structural dynamics is β, since it is sufficient to know the *percentage* of damping rather than the actual values of C or Ccr. Let the solution of Eq. (10) be in the form:

$$x = e^{st}$$

therefore,
$$\dot{x} = se^{st}$$

$$\ddot{x} = s^2e^{st}$$

and substituting, $e^{st}\ (s^2 + 2\beta w \bullet s + w^2) = 0$ (11)

The roots are

$$S_{1,2} = \frac{-2\beta w \mp \sqrt{4\beta^2w^2 - 4w^2}}{2}$$

$$= (-\beta \mp \sqrt{(\beta^2 - 1)w)} \tag{12}$$

It is seen that if $\beta = 1$ (the case of critical damping), of if $B > 1$ (called overdamped case), there is no vibration. The only roots of interest are those when $\beta < 1$ and substituting and introducing the two constants:

$$x = Ae^{(\beta + \sqrt{(1 - \beta^2 i)wt)}} + Be^{(-\beta - \sqrt{(1 - 3^2 i)wt)}} \tag{13}$$

Introducing $w_d = \sqrt{1 - \beta^2}\ v$, Eq. (13) is simplified into

$$x = e^{-\beta wt(C_1 \cos w_d t + C_2 \sin w_d t)} \tag{14}$$

If $\beta = 0$, Eq. (14) reduces to Eq. (9) with $A = C_1$ and $B = C_2$. The term w is called the natural circular frequency (radius/sec). Other definitions are

$$\text{Natural Period} = T = \frac{2\pi}{w}\ \text{(seconds)} \tag{15}$$

$$\text{Natural frequency} = f = \frac{1}{T}\ \text{(cycles/sec).} \tag{16}$$

Let's take the undamped free case of Eq. (9) which is simpler to illustrate. If the initial displacement and initial velocity of the system are, respectively, x_0 and x_0, we differentiate Eq. (9) once with respect to time to get the expression for velocity, and differentiate again to obtain the expression for acceleration. Substituting the values of the initial conditions, we have two equations which enable us to solve for the constants A and B, and

$$x + x_0 \cos wt + \frac{\dot{x}_0}{w} \sin wt \qquad (17)$$

In the same manner, the damped free case of Eq. (14) gives

$$x = e^{-\beta wt} \left(x_0 \cos w_d t + \frac{\dot{x}_0}{\frac{w}{\sqrt{1-\beta^2}}} + \beta x_0 \sin w_d t\right) \qquad (18)$$

The graphical representation of Eq. (17) is shown in two components in Fig. 7:

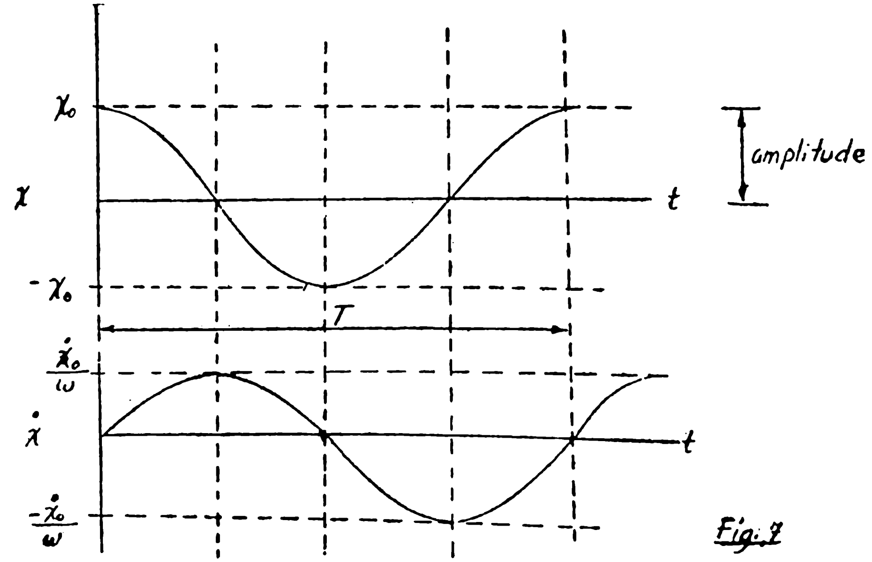


Fig. 7

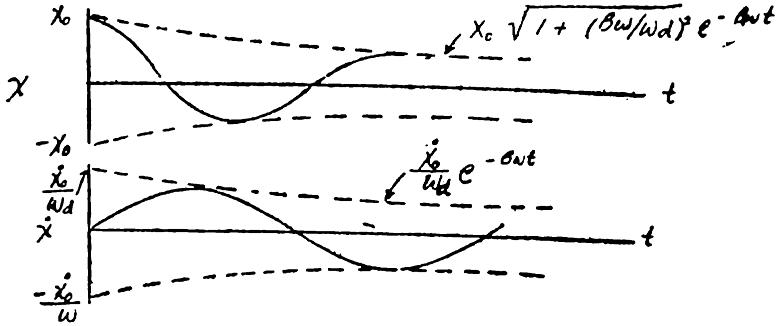


Fig. 8

Eq. (18) is represented graphically by Fig. 8 above.

With regard to the differentiationn between w and d_d, observations indicate that a typical structure have between 5 to 10 per cent of critical damping and for this matter, it may be concluded that the decrease of natural circular frequency due to damping may reasonably be ignored.

It is convenient to get a measure of damping by noting the decrease in amplitude between two successive peaks of the amplitude of vibration. It is easy to show that

$$\frac{x_k}{x_{k+1}} = e^{\beta w T_d} = e^{\delta} \text{ where } T_d = \frac{W_d}{2\pi} = \frac{2\pi}{W_d} \qquad (19)$$

or the ratio between any two peaks:

$$\frac{x_k}{x_{k+n}} = e^{n\delta} \qquad (20)$$

the parameter δ is called the logarithmic decrement.

The solution of the *forced* vibration case involves looking for the particular solution in addition to the homogeneous solution already determined. F(t) is a forcing function usually represented by a graph, and can also be written $F(t) = F_c \cdot f(t)$ where f(t) is a dimensionless time function. To illustrate the method, we assume F(t) given by Fig. 9, which is best described as a suddenly applied load with infinite time or with a time duration.

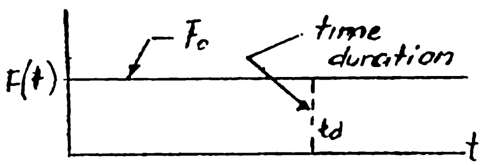


FIGURE 9

Assume an undamped forced vibration such that F(t) is given by Fig. 9 and of infinite duration. The equation of motion is

$$m\ddot{x} + kx = F(t) = F_o \qquad (21)$$

The complementary solution of the above equation is given by Eq. (9) as discussed before. The particular solution is $X_p = \frac{F_0}{K}$ and the solution is

$$X = x_c + x_p = A \cos wt + B \sin wt + \frac{F_0}{K} \qquad (22)$$

Differentiating once, and then again, with respect to time to get expressions for velocity and acceleration, and incorporating initial conditions, we find

$$A = x_0 - \frac{F_0}{K} \text{ and } B = \frac{x_0}{w}$$

Substituting back, we get the response

$$x = \underbrace{x_0 \cos wt + \frac{x_0}{w} \sin wt}_{\text{homogeneous solution}} + \underbrace{\frac{F_0}{K} (1 - \cos wt)}_{\text{due to forcing function}} \qquad (23)$$

As seen from the above example, the determination of the particular solution is the key to the solution and this can readily be done only for the simpler pulses. If the forcing function curve has irregular (or also discontinuous) shape, it is important to 1_{oo} for other methods.

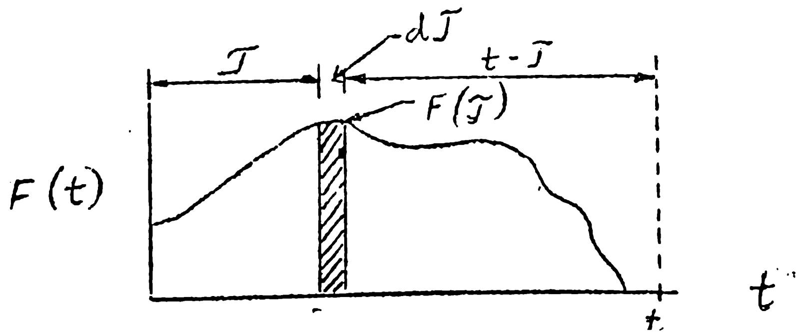


Fig. 10

Fig. 10 shows a general forcing function curve, and the parameter is made the independent time variable. It is required to get the response of an undamped SDF system due to the effect of an increment of load F (T) shown cross-hatched in Fig. 10. Since F(T) is acting in a very short interval of time, the impulse i of the load is F(T) • dT which is equal to the momentum mx_i or

$$x_i = \frac{F\ (T)\ \bullet\ (dT)}{m} \qquad (24)$$

where X_i is the velocity at time T, and may be considered as initial velocity imparted to the system at rest. The displacement at a later time t, due to this single increment of impulses for the undamped case is given by Eqn (17). We make X_i the initial velocity, and initial displacement equals zero,

$$dx = \frac{F(T) \cdot dT}{mw} \sin w\ (t-T) \tag{25}$$

It should be clear that as soon as the impulse acts, there will be an "instantaneous" change in velocity but the initial displacement X_o is zero so that the cosine term drops out in Eqn. (17). Eqn. 25 represents the contribution to the response due to an increment of impulse. Summing up and noting that $w^2m = K$,

$$X = \frac{F\ (T)}{K} w \sin w\ (t-T)\ dT, \text{ or}$$

$$X = \frac{}{K} w \int_{t_1}^{t_2} f\ (T) \sin w(t-T)\, dT \tag{26}$$

It is clear that $\frac{Fo}{K}$ is the static deflection if Fo is the static load and this ratio is sometimes written X_{st}. The limits t_1 and t_2 refer refer to the inclusive times during which the forcing function is acting. The response given by Eqn. (26) is a product of two factors, namely,

$$X = X_{st}. \qquad \text{Duhamel Integral}$$

For a suddenly applied constant load Fo, the Duhamel Integral is

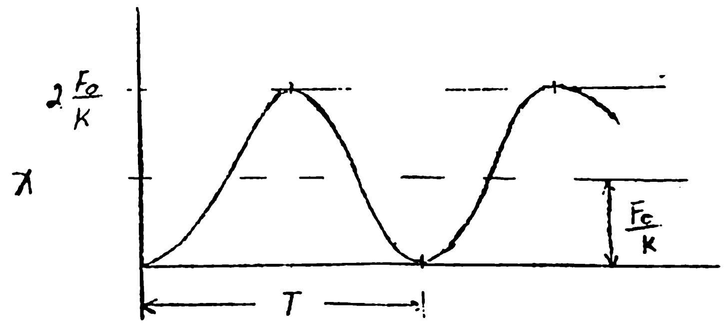
$$w \int_0^t \sin w(t - T)\, dT = w(\frac{1}{w_t}) \int_0^t \sin w(t-T)\ dT\ (-w)$$

$$= -[-\cos(wt-wT)] \quad = (1-\cos wt) \tag{27}$$

which is identical to the solution given by Eqn. (23). A similar derivation may be done for the damped case which gives the following result for the Duhamel, or sometimes called the convolution, Integral:

$$\frac{w^2}{wd} \int_{t_1}^{t_2} e^{-\beta\omega\ (t-T)} f(T) \sin w_d(t - T)\, dT \tag{28}$$

It is interesting to know how the response curve compares with the free vibration case:



It can be observed that the curve is very similar to the previous solution for the free vibration case shown on Fig. 7. The only difference is that the axis is shifted by $X_{st} = Fo/K$ so that the maximum displacement is exactly *twice* the displacement that would occur if the load Fo were applied statically. The evaluation of the Duhamel Integral introduces the concept of Dynamic Load Factor, or Amplification Factor, which is an non-dimensional factor to be multiplied to X_{st} in order to determine the maximum dynamic amplitude of vibration.

Suppose that the suddenly applied constant load has a duration t_d as shown in Fig. 9. The equations of motion are (for the undamped case)

$$m\ddot{X} + Kx = Fo \qquad 0 \leq t \leq t_d$$

$$m\ddot{X} + Kx = 0 \qquad t_d \leq t$$

For the first of the above equations, we do have the solution assuming initial displacement and initial velocity are zero,

$$X = \frac{Fo}{K}(1-\cos wt) \qquad X_d = \frac{Fo}{K}(1-\cos wt_d)$$

$$\dot{X} = \frac{Fo}{K} w \sin wt \qquad \dot{X}_d = \frac{Fo}{K} w \sin wt_d$$

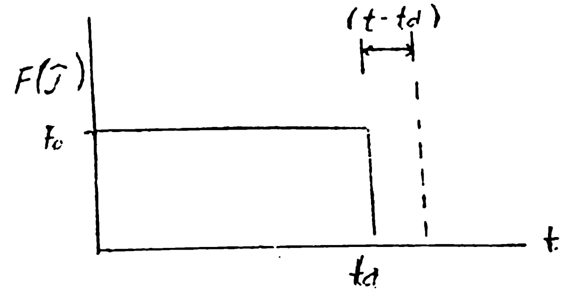


Fig. 12

At time t_d, we may now substitute the initial conditions X_d and $\dot{X}_d$ and give the solution for Fig. 12.

The free vibration case. At time $t > t_d$, therefore, the response is

$$X = X_d \cos w(t-t_d) + \frac{\dot{X}_d}{w} \sin w(t-t_d) \tag{29}$$

which can be simplified as

$$X = \frac{Fo}{K} [\cos w(t-t_d) - \cos wt] \tag{30}$$

By Duhamel Integral,

$$X = \frac{Fo}{K} w \int_0^{t_d} \underbrace{f(T) = 1}_{\text{Duhamel}} \sin w(t-T)\, dT$$

$$= \frac{Fo}{K} w\left(\frac{1}{w}\right) \int_0^{t_d} \sin w(t-T)\ (-w)\ dT$$

$$= - [\cos w(t-T)]_0^{t_d} = \frac{Fo}{K} [\cos w(t-t_d) - \cos wt] \tag{31}$$

which gives the same result as Eqn. (JC). In none-dimensionalized form,

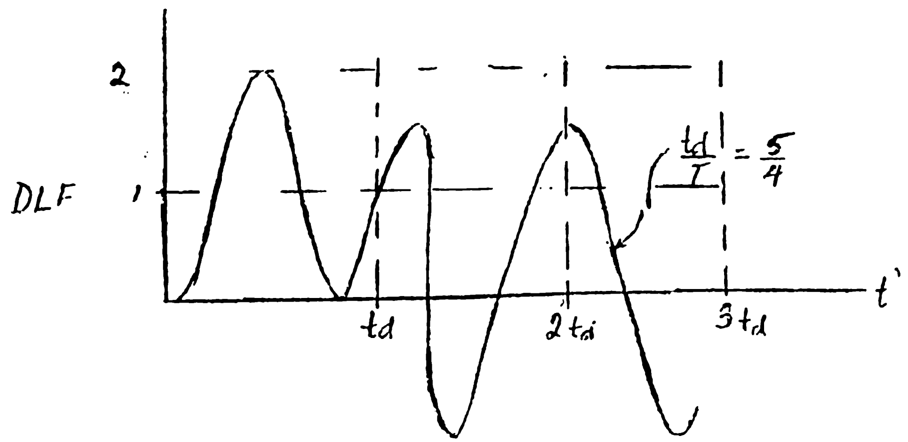


FIG. 13

Eqn. (30) or (31) is graphed in Fig. 13 above for $T_d/T = 5/4$. One important feature of the graph is that the DLF (dynamic load factor) is 2, as found earlier, for $0 < t \leq t_d$, which is a forced vibration case. For $t \geq t_d$, we have free vibration (the forcing function has ceased to act) and the amplitude becomes less with the corresponding decrease in the DLF.

It is seen that the Duhamel Integral is indeed a powerful tool in the evaluation of responses due to any shape of forcing function. However, if the function cannot be expressed mathematically conveniently, the computation of the response can be well solved by numerical methods.

There are many force pulses which may be useful for exercises and which may represent the approximate pulse. However, of particular interest to earthquake engineering is the case of support motion. The model of the SDF system involving this class of problems is shown in Fig. 14. Suppose that the system is subjected to

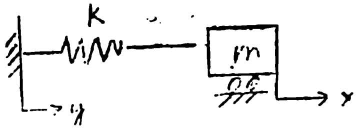


FIG. 14

support motion y defined by a displacement function $y = y_0 \, . \, f(t)$ where y_0 is some arbitrary magnitude of support displacement. The equation of motion is

$$m\ddot{x} + K(x\text{-}y) = 0$$

$$\text{or } m\ddot{x} + Kx = Ky = (Kyo) \; . \; f(t) \qquad (32)$$

For a suddenly applied constant support displacement, the right hand side of Eqn. (32) becomes Kyo) since $f(t) = 1$. The form of Eqn. (32) then is the same as that of Eqn. (21) — and the solution is readily written:

$$x = y_0 \; (1\text{-cos } wt)$$

If the relative displacement is defined as $u = x\text{-}y$,

$$u = y_0 \; (1\text{-cos } wt) — y_0 = y_0 \cos wt \qquad (33)$$

The force in the spring is K_u and the negative sign on the right hand side of Eqn. (33) simply means that the spring is initially in compression when y_0 is positive.

When the input is support acceleration rather than support displacement, it is convenient to change the variable into *relative* displacement, thus

Let $u = x - y$ as before, then $\ddot{u} = \ddot{x} - \ddot{y}$ (34)

The equation of motion is

$$m\ddot{x} + (K)(x-y) = 0$$

$$m(\ddot{u} + \ddot{y}) + Ku = 0$$

$$\text{or } m\ddot{u} + Ku = m\ddot{y} = -m\ddot{y}_0 \,.\, f(t) \qquad (35)$$

Eqn. (35) is identical to those for forcing functions if F_0 is replaced by $-m\ddot{y}_0$ and the general solution for the relative motion u is

$$u = \frac{m\ddot{y}_0\, w}{K} \; f(\tau) \; \sin w(t - \tau) d\tau \qquad (36)$$

There are useful remarks regarding Eqn. (32) and Eqn. (35). The former does not determine the spring force, which represents member force, *directly*, whereas the latter (Eqn. 35) does. Besides most data on ground motion recorded by instruments are in terms of acceleration. If damping is considered and support motion is in terms of acceleration, the equation of motion is

$$m\ddot{x} + c(\dot{x}-\dot{y}) + K(x-y) = 0$$

$$\text{or } m\ddot{u} + c\dot{u} + Ku = m\ddot{y}_0 \,.\, ft(t) \qquad (37)$$

which is similar in form to Eqn. (6) and solution is available by merely replacing F(t) or F_0 . f(t) by $-m\ddot{y}_0$. f(t) on the right side. If support motion cannot be expressed in terms of acceleration, it becomes necessary to include *both* displacement and velocity of the support, thus:

$$m\ddot{x} + c(\dot{x}-\dot{y}) + K(x-y) = 0 \text{ as in Eqn. (37)}$$

We write

$$m\ddot{x} + C\dot{x} + Kx = c\dot{y} + Ky = c\dot{y}_0\, f(t) + Ky_0\, f(t)$$

MULTI-DEGREE OF FREEDOM SYSTEMS; THE MODAL ANALYSIS

When the configuration can no longer be defined by a single coordinate, then we have a MDF (Multi-degree-of-freedom) System. Since the formulation of the problem is the same regardless of the number of degrees of freedom beyond one, we will discuss two-degree-of-freedom systems for simplicity. Fig. 15 shows one system having two degrees of freedom. The masses are once again assumed to be concentrated at the floor levels and the stiffness of the two columns at each floor. The frame of Fig. 15 may be idealized by the model also shown on the right of the frame in the same figure. The free body diagrams of the two masses are shown in Fig. 16 below. The Equations:

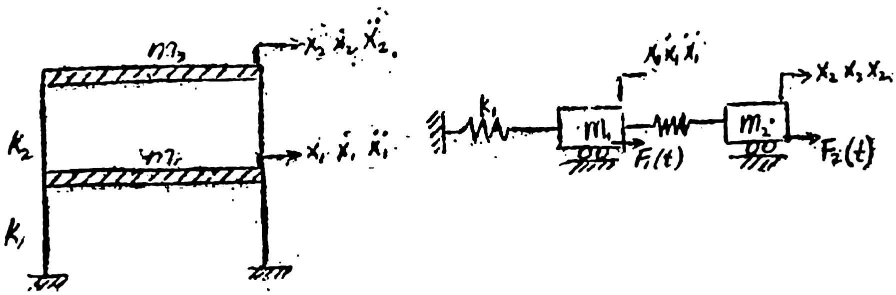
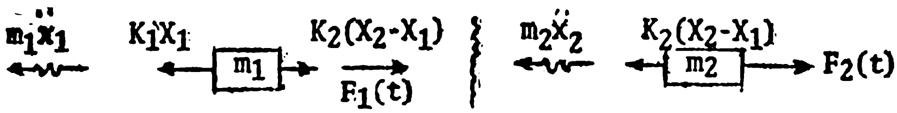


FIG. 15



of motion are readily written:

$$m_1\ddot{x}_1 + K_1X_1 = K_2(X_2\text{-}X_1) = F_1(t)$$

$$m_2X_2 + K_2\ (X_2\text{-}X_1) = F_2(t) \qquad (38)$$

Eqns. (38) can be conveniently written in matrix form:

$$\begin{bmatrix} m_1 & o \\ o & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} (K_1+K_2) & K_2 \\ -K_2 & -K_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \qquad (39)$$

or more compactly,

$$[M]\ \{X\} + [K]\ \{X\} = \{F(t)\} \qquad (40)$$

Equations of motion of higher degree of freedom systems can also be generated in the same manner, but the compact form will be identical to Eqn. (40) which is the general form of the matrix equation of motion for any multi-degree of freedom system. [m] is called the mass matrix [K] is the stiffness matrix, { X } is the acceleration vector, { X } the displacement vector and the F(t) the forcing function vector. For discrete systems, the forms of the [m] and [K] matrices depends upon the *coupling* of the equations. Certainly it must be realized that the [K] is always a symmetric matrix because of Maxwell's Reciprocal Theorem.

To understand the meaning of the elements of the stiffness matrix we write for examples

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{22} & K_{33} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

where the K's are stiffness coefficients". For example, the force on P_1 due to a displacement X_2 is merely $P_1 = K_{12}X_2$, etc. If damping is considered, then the matrix equation of motion is

$$[m]\{\dot{X}\} + [c]\{\dot{X}\} + [K]\{X\} = \{F(t)\} \qquad (41)$$

which is exactly in the same form as Equ (6) for SDF systems. The new terms to be defined are [c] which is called the damping matrix, and { x } which is the velocity vector. Since the matrix equation of motion, Equ (41) or Equ (40) represents, in compact form, independent differential equations of motion, the sizes of the matrices and vectors correspond to the number of these independent equations, which is also equal to the number of degrees of freedom of the system.

The solution of Equ (40) or Equ (41) involves the simultaneous solution of n equations to solve for a unknowns, where n is the number of degrees of freedom.

It is appropriate to introduce the concept of *normal modes of vibration.* A system is said to have exactly the same number of normal modes as degrees of freedom. Associated with each normal mode is a *natural frequency* and a *characteristic shape.* The distinguishing feature of a normal mode is that the system could, under certain conditions, vibrate *freely* in that mode alone. During such vibration, the ratio between any two displacements is constant with time. These ratios define the characteristics shape for that mode alone. An extremely important fact, which is the basis of the *Modal Analysis* of multi-degree of fredom systems is that the complete motion of the system may be obtained by superimposing the independent motions in the individual modes.

The equations of motion in matrix form of a sytsem having n masses and n degrees of freedom but no external forces is

$$[M]\{\ddot{X}\} + [K]\{X\} = + \{X\} = \{0\} \qquad (42)$$

The undamped case is taken because in the normal modes, the masses vibrate freely.

$$\text{Let}\left\{X\right\} = \left\{\bar{X}\right\} \sin wt, \quad \therefore \left\{\ddot{X}\right\} = -w^2\left\{\bar{X}\right\}\sin wt$$

where $\{X\}$ is called the amplitude vector and w is a constant. Substituting equations (43) into Equ. (42)

$$-W^2 \left[m\right]\left\{\bar{X}\right\} \sin wt + \left[K\right]\left\{\bar{X}\right\} \sin wt = \left\{0\right\}$$

or placing in standard form,

$$\left\{\left[K\right] - w^2\left[m\right]\right\}\left\{\bar{X}\right\} = \left\{0\right\}$$

which represents a "generalized eigen value problem", and the formulation of Equ. (44) is called Stiffness Formulation. The inverse formulation, called the Flexibility formulation, is derived by premultiplying both sides of Equ. (42) by [F] which is the flexibility matrix. This matrix [F] is the inverse of the stiffness matrix [K]. Writing therefore,

$$[F]\ [m]\ \{\ddot{X}\} + \{X\} = \{O\} \qquad (45)$$

Making [F] [m] [D] called the dynamical matrix, the standard form in the flexibility formulation reduces to

$$\left\{\left[D\right] - \frac{1}{w^2}\left[I\right]\right\}\left\{\bar{X}\right\} = \left\{0\right\} \qquad (46)$$

which is a "regular eigenvalue problem."

Expanding Equ. (44), we get n homogeneous equations which can be solved only for the *relative* values of the X's. Recalling Cramer's Rule for solving such equations non-trivial values of x exist only if the determinant of the coefficients is equal to zero. For an n-degree of freedom sytsem.

$$\begin{vmatrix} (K_{11} - m_1 W^2) & K_{12} & \cdots & kin \\ K_{21} & (K_{ii} - m_i W^2) & & \vdots \\ \vdots & & \ddots & \\ Knl \cdots & & & (K_{nn} - M_n W^2) \end{vmatrix} = O \quad (47)$$

The expansion of the above determinant leads to the so-called "characteristic equation" of the form.

$$(W^2)^n + C_1 (W^2)^{n-1} + \ldots C_i (W^2)^{n-i} + \ldots Cn = O \qquad (48)$$

The largest root gives the highest frequency or the frequency of the highest mode. A similar procedure can be followed to solve Equ (46) and when this is done, the largest root gives the lowest frequency or the frequency of the fundamental mode.

If the system shown in Fig. 15 is made to vibrate freely that is F_1 (t) $=$ F_2 (t) $=$ O, following the stiffness approach formulated by Equ. (47) gives the following characteristics equation:

$$(W^2)^2 - \left[\frac{K_1 + K_2}{m_1} + \frac{K_2}{m_2}\right] (W2) + \frac{K_1}{m_1} , \frac{K_2}{m_2} = O \qquad (49)$$

If $m_1 = M_2 = M$, and $K_1 = K_2 = K$, the roots are

$$W_1^2 = O,38 \frac{K}{M} \text{ , or } W_1 = 0.616 \sqrt{\frac{K}{M}} \text{ rad/sec} \qquad (50)$$

$$W_2^2 = 2.62 \frac{K}{M} \text{ , or } W_2 = 1.616 \sqrt{\frac{K}{M}} \text{ rad/sec}$$

Substituting back these W^2 into Equ (44), we find that

$$\{X_1\} = \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix} \quad \text{and} \quad \{\bar{X}_2\} = \begin{Bmatrix} 1 \\ -.62 \end{Bmatrix} \qquad (50a)$$

which are the mode shape vectors commonly written as Θ_1 and Θ_2. These define the characteristic shapes of the two modes.

The procedure outlined above is easily applied to 2 or 3 degree of freedom system, but this "direct" method is cumbersome if the number of degrees of freedom becomes big. Two methods which are easily geared to the computer are the Matrix Iteration Method, proposed by Stodolla-Vianelo, and Holzer's Method. The so-called "sweeping technique enables the engineer to sweep out succeeding mode shapes thus eliminating the simultaneous substitution to solve for the next mode shape.

An important property of mode shape vectors is that of *orthogonality* of the modes which means that the mode shape vectors any two modes, i and j for instance form a product such that

$$\{\Theta i\}^t [M] \{\Theta_j\} = = \{O\}$$

THE MODAL ANALYSIS assumes that we may represent the response of the sytsem as a combination of the responses of the natural modes. For the two-degrees-of-freedom system, the response in matrix form is

$$\begin{Bmatrix} X_1(t) \\ X_2(t) \end{Bmatrix} = A_1\{\theta_1\}\cos w_1 t + B_1\{\theta_1\}\sin w_1 t + A_2\{\theta_2\}\cos . w_2 t + B_2\{\theta_2\}\sin w_2 t \quad (51)$$

There are 4 constant sin the A's and B's. $\{\Theta_i\}$ is the mode shape vector in mode i and the corresponding circular frequency is W_i. If the initial conditions are now given by two vectors $\{X_0\}$ and $\{X_0\}$, which are the initial velocity and initial displacement vectors, respectively,

$$A_i = \frac{\{\theta_i\}^t[m]\{X_o\}}{\{\theta_i\}^t[m]\{\theta_i\}} \quad \text{and} \quad B_i = W_i\frac{\{\theta_i\}^t[m]\{\dot{x}_o\}}{\{\theta_i\}^t[m]\{\theta_i\}} \quad (52)$$

Equs. (52 enable us to write the response of a multi-degree of freedom system by superimposing the effects of all the modes as follows:

$$\{X(t)\} = \sum_{i=1}^{\text{modos}} A_i\{\theta_i\}\cos W_i t + \sum_{i=1}^{\text{modes}} B_i\{\theta_i\}\sin w_i t \quad (53)$$

We have already seen that the matrix equations of motion is a set of n simultaneous differential equations where n equals to the number of degrees of freedom. These equations are obviously *coupled.* If we can find a way to uncouple these solutions, we may be able to solve the equations one by one, since in this way each equation will yield only one unknown. Fortunately, it is possible to transform the system of equations as formulated in "generalized coordinates" thus

$$[M]\{\ddot{X}\} + [K]\{X\} = [P](t) \quad (54)$$

(and if damping is considered, the left side will have another matrix factor + [c] $\{(x)\}$ into another set of coordinates called normal, or principal coordinates.