PLANE STRUCTURE-MEDIUM INTERACTION UNDER EARTHQUAKE DISTURBANCE

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INTRODUCTION

In almost all methods proposed for the dynamic analysis of multistorey rigid frames subjected to an earthquake-type disturbance, no consideration has been formalized with respect to the interaction between the structure and its surrounding medium. A common practice assumes that the structure is attached to a perfectly rigid foundation and disregards any local influence of the medium on the resulting structural response. Thus, the dynamic disturbance is frequently oversimplified by replacing loads with their static equivalents, either by assuming lateral loads at every floor level or by subjecting the structure to an overall effect of a total base shear. It is understandable that assumptions must be selected so as to simplify the mathematical solution, but there is growing evidence through actual observations that other parameters of the coupled structure-medium interaction must be studied. This dynamic coupling between the structure and its foundation is a function of the energy exchange between them and it is believed that the natural period of vibration and the resulting response of the structure may be affected significantly by this fact.

Discussion of the theory of elastic waves in solids is beyond the scope of this paper. However, a basic knowledge of how the dynamic disturbance is propagated in solids is essential. It is therefore sufficient to state that when an elastic medium is deformed according to some input function, such as displacement, velocity or acceleration, adjacent particles are displaced, and as time passes, more and more particles are included in the disturbed region. The net result is the propagation of an elastic stress wave through the medium. Since particles will certainly not develop infinite displacements, such of them is expected to move within a bounded region. The net result is the propagation of an elastic stress wave through the medium. Since particles will certainly not develop infinite displacements, each of them is expected to move within a bounded region. These individual oscillations, however, are not independent of each other and as a result, motion relationships between adjacent particles are collectively known as a coupled condition.

In an isotropic and homogeneous medium, longitudinal and transverse waves are normally produced. These waves travel at known velocities in essentially a "free-field" condition as long as the medium remains semi-infinite in extent. However, when either wave type reaches a boundary or a discontinuity, waves of both types are generated by reflection and by refraction.

In this study, the structure is assumed to be of uniform, homogeneous isotropic material and is idealized as a lumped-mas model. The resistances in generalized coordinates are derived by a reduction scheme of the singular stiffness matrix previously assembled by conventional structural methods. A useful simplication neglects axial distortions and the equations of motion of the lumped masses in the generalized coordinates, together with the generalized stiffness matrix and the imposed boundary conditions, form the mathematical formulation of the problem.

The medium is likewise idealized as a lumped-parameter model, the dynamic analysis of which leads to a system of centered finite difference equations which have been shown to be mathematically consistent with the corresponding equations of the continuum. This approach basically transforms the continuum, which possesses an infinite number of degrees of freedom, to one which has a finite number of degrees of freedom. The structure-medium coupled solution results in recursive equations which are integrated numerically. The unknowns of the problem include all the components of stress at every assumed "stress point" and the components of acceleration, velocity and displacement at every "mass point" for each successive time interval.

Inherent in any numerical procedure is the fact that solutions can, at best, be only approximate. This is mainly due to the fact that the real continuous medium has been replaced by a discrete model in order to reduce the number of degrees of freedom to a reasonable finite number. Discussion of theoretical round-off errors due to this discretization, and also the stability and convergence requirements of the present numerical scheme are also beyond the scope of this paper.

The purpose of the investigation is to evaluate the influence of the structure-medium interaction on the response of a plane rigid frame subjected to an earthquake-type disturbance. Three interrelated problems are solved as follows: (1) The coupled structure-medium system, (2) The same structure on a rigid foundation subjected to the "free-field" effect of an input emanating from a point away from the structure, and (3) The structure on a rigid foundation subjected to direct base disturbance which is the input itself.

Constitutive Relations Of The Solid Medium

The general theory of behavior of elastic solids is extensively developed and discussed in many texts on elasticity. The Dynamical equations of motion of an infinitesimal element of volume are derived on the basis of the theory of stress. The most general form in which the relation among stress components and strain components may be expressed symbolically is given by:

$$\{\sigma\} = [C]\{\epsilon\} \tag{1}$$

where each of the stress and strain vectors consists of six components, and [C] is a 6 x 6 matrix of elastic constants. For an isotropic material, only two independent constants are required for the complete definition of [C]:

$$\lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} \quad \text{and} \quad G = \frac{E}{2(1+\upsilon)} \tag{2}$$

where E is the modulus of elasticity and v is Poisson's Ratio of the medium.

In the present case, a typical plane rigid frame, lying in the x-y plane is considered one of many which form a relatively long building whose axis lies along the z-direction. For these reasons, plane strain condition in a two-dimensional case is assumed, and Equation (1), when expanded, becomes:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \mathcal{T}_{xy} \end{Bmatrix} = \frac{E}{(1+\upsilon)(1-2\upsilon)} \begin{bmatrix} 1-\upsilon & \upsilon & \upsilon & 0 \\ \upsilon & 1-\upsilon & \upsilon & 0 \\ \upsilon & \upsilon & 1-\upsilon & 0 \\ 0 & 0 & 0 & \frac{1-2\upsilon}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ V_{xy} \end{Bmatrix} \tag{3}$$

or simply,

$$\{\sigma\} = [D]\{\epsilon\} \tag{3a}$$

The Medium Model

In the two-dimensional x-y plane, the differential equations of motion of an infinitesimal element of volume are:

$$\frac{\partial \sigma_x}{\partial X} + \frac{\partial \tau_{xy}}{\partial Y} = P \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial \tau_{xy}}{\partial X} + \frac{\partial \sigma_y}{\partial Y} = P \frac{\partial V^2}{\partial t^2} \qquad (4)$$

where ρ is the mass density of the medium, and the stresses are defined in the usual convention. Substituting the strain-displacement relations and the constitutive equations of elasticity into the above Equations (4), we have:

$$(\lambda + G)\frac{\partial e}{\partial X} + G\Delta^2 u = P \frac{\partial^2 u}{\partial t^2}$$
$$(\lambda + G)\frac{\partial e}{\partial Y} + G\Delta^2 V = P \frac{\partial^2 V}{\partial t^2} \qquad (5)$$

where Δ^2 is an operator defined as

$$\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \quad \text{and} \quad e = \varepsilon_x + \varepsilon_y$$

By proper mathematical manipulation, Equations (5) may be shown to yield the following:

$$(\lambda + 2G)\ \Delta^2 e = P \frac{\partial^2 e}{\partial t^2}$$
$$G\Delta^2 \overline{W}_z = P \frac{\partial^2 \overline{W}_z}{\partial t^2} \qquad (6)$$

where $\overline{W}_z$ represents rotation about the z-axis. Equations (6) are clearly wave equations which give, respectively, the dilatational and transverse wave velocities:

$$C_1 = \sqrt{\frac{\lambda + 2G}{P}} \quad \text{and} \quad C_2 = \sqrt{\frac{G}{P}} \qquad (7)$$

The basic generalized medium element is taken as a stress point '0' with its associated four mass points numbered 1, 2, 3 and 4 as shown in Fig. 1. In this element, strains are defined by the displacements of the four masses, whereas stresses are defined only at is represented in non-cartesian reference. Since the medium has been assumed to be uniform, the mass lumped at a point is the summation of the contributions of tributary volumes (or areas for planar conditions) as shown in Fig. 2, which also shows the ordering of the four masses. The volume tributary to Mass 1, for example, is proportional to:

$$V_1 = KA_1 = K\frac{A}{4}\left[\frac{1}{2} + \frac{\Delta^2}{\Delta\bar{x}}\right] \tag{8}$$

where A is the area bounded by the four masses. With further reference to Fig. 1, the following are defined:

x, y and z as the rectangular cartesian global reference

$\bar{x}$, $\bar{y}$ and $\bar{z}$ as the non-orthogonal local coordinates.

These two sets of reference axes are related by the following matrix equation:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \cos\beta & 0 \\ \sin\alpha & \sin\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} \tag{9}$$

or inversely:

$$\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} = \frac{1}{\sin(\beta-\alpha)} \begin{bmatrix} \sin\beta & -\cos\beta & 0 \\ -\sin\alpha & \alpha & 0 \\ 0 & 0 & \sin(\beta-\alpha) \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

Assuming small-displacement theory, strains may be expressed in terms of the partial derivatives of displacements. In matrix form, the relation between incremental strains and incremental displacements is:

$$\begin{Bmatrix} \Delta e_x \\ \Delta e_y \\ \Delta e_z \\ \Delta v_{xy} \end{Bmatrix} = \frac{1}{\sin(\beta-\alpha)} \left[\beta_1 \,\vdots\, -\beta_2 \,\vdots\, -\beta_1 \,\vdots\, \beta_2\right] \begin{Bmatrix} \Delta u_1 \\ \Delta V_1 \\ \Delta u_2 \\ \Delta V_2 \\ \Delta u_3 \\ \Delta V_3 \\ \Delta u_4 \\ \Delta V_4 \end{Bmatrix} \tag{11}$$

where:

$$[\beta_1] = \begin{bmatrix} \sin\beta/\Delta\bar{x} & 0 \\ 0 & -\cos\beta/\Delta\bar{x} \\ 0 & 0 \\ -\cos\beta/\Delta\bar{x} & \sin\beta/\Delta\bar{x} \end{bmatrix}; \quad [\beta_2] = \begin{bmatrix} \sin\alpha/\Delta\bar{y} & 0 \\ 0 & -\cos\alpha/\Delta\bar{y} \\ 0 & 0 \\ -\cos\alpha/\Delta\bar{y} & \sin\alpha/\Delta\bar{y} \end{bmatrix}$$

or simply:

$$\{\Delta\epsilon\} = 1/\sin(\beta-\alpha)\,[B]\,\{\Delta u\} \qquad (13)$$

By the Principle of Virtual Work, it can be shown:

$$\{\Delta P\} = \frac{1/2\ \Delta\bar{x}\Delta\bar{y}\ \sin(\beta-\alpha)}{\sin^2(\beta-\alpha)}\,[B]^t[D][B]\,\{\Delta u\}$$

$$= [Ke]\,\{\Delta u\} \qquad (14)$$

in which $\{\Delta P\}$ is the vector of incremental nodal forces, $[D]$ is the matrix defined in Equation (3a), and $\frac{1}{2}\ \Delta\bar{x}\Delta\bar{y}\ \sin(\beta-\alpha)$ represents the volume of the basic medium element. Also, in the above Equation (14), $[Ke]$ is clearly the stiffness matrix of the non-orthogonal lumped-parameter medium element in global reference.

It is convenient to express the triple matrix product $[B]^t[D][B]$ by representing the $[B]$ matrix by its submatrices defined in Equation (12):

$$[Ke] = \frac{1/2\ \Delta\bar{x}\Delta\bar{y}}{\sin(\beta-\alpha)} \left[\begin{array}{c|c|c|c} B_1^tDB_1 & -B_1^tDB_2 & -B_1^tDB_1 & B_1^tDB_2 \\ \hline -B_2^tDB_1 & B_2^tDB_2 & B_2^tDB_1 & -B_2^tDB_2 \\ \hline -B_1^tDB_1 & B_1^tDB_2 & B_1^tDB_1 & -B_1^tDB_2 \\ \hline B_2^tDB_1 & -B_2^tDB_2 & -B_2^tDB_1 & B_2^tDB_2 \end{array}\right] \qquad (15)$$

The Structure Model

The structure modeled for this study is a three-storey single-bay plane rigid frame with basement. Discretization of the structure locates the lumped masses at the actual joints as well as at convenient positions around the basement where the structure interacts with the medium as shown in Fig. 5. The force and displacement vectors are thus defined at these masses in the generalized coordinates as shown in Fig. 14, and these vectors are related by the generalized stiffness matrix of the structure. It is further assumed that the elements are prismatic and uniform and that the structure remains elastic throughout the entitre time of interest.

In addition to boundary constraints along the structure-medium interface, other simplifications are made which make the calculations shorter. One of these assumptions is to neglect axial distortions, which makes the number of constraint equations equal to the number of structural elements. The shears and monents at the two ends of an element i–j are related to the corresponding displacements by:

$$\begin{Bmatrix} \bar{V}_i \\ \bar{M}_i \\ \hdashline \bar{V}_j \\ \bar{M}_j \end{Bmatrix} = \frac{2EI}{L^2} \left[\begin{array}{cc:cc} \frac{6}{L} & 3 & -\frac{6}{L} & 3 \\ 3 & 2L & -3 & L \\ \hdashline -\frac{6}{L} & -3 & \frac{6}{L} & -3 \\ 3 & L & -3 & 2L \end{array}\right] \begin{Bmatrix} \bar{U}_i \\ \bar{V}_i \\ \hdashline \bar{U}_j \\ \bar{V}_j \end{Bmatrix} \qquad (16)$$

where the quantities, of course, are in member (or local) coordinates. In order to assemble the stiffness matrix, force and displacement vectors of Equation (16) are transformed into the global reference by means of a rotation matrix, and since force and displacement vectors transform identically in rotation, we have:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} -\sin\alpha & 0 \\ \cos\alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{y} \\ \bar{z} \end{Bmatrix} \text{ and } \begin{Bmatrix} F_x \\ F_y \\ M \end{Bmatrix} = \begin{bmatrix} -\sin\alpha & 0 \\ \cos\alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{F}_y \\ \bar{M} \end{Bmatrix}$$

$$\text{or} \quad \{x\} = [R]\{\bar{y}\}$$
$$\text{and} \quad \{F_x\} = [R]\{F_y\} \tag{17}$$

Equation (16) in global coordinates become:

$$\begin{Bmatrix} F_{xi} \\ F_{yi} \\ M_i \\ \hdashline F_{xj} \\ F_{yj} \\ M_j \end{Bmatrix} = \left[\begin{array}{c:c} RK_{ii}R^t & RK_{ij}R^t \\ \hdashline RK_{ji}R^t & RK_{jj}R^t \end{array}\right] \begin{Bmatrix} u_i \\ V_i \\ \theta_i \\ \hdashline u_j \\ V_j \\ \theta_j \end{Bmatrix} \tag{18}$$

The structure stiffness matrix is assembled by direct summation of the contributions of all the elements comprising the structure, and without any external constraints as yet provided, this matrix is singular. If {Q} is a set of generalized nodal forces corresponding to the set of generalized nodal displacements {q}, the generalized stiffness matrix relating these two sets of vectors [K*] is square and of the order equal to the number of generalized coordinates of interest. In this study, the number of generalized coordinates, as shown in Fig. 14, for the coupled and the uncoupled solutions, respectively, are 11 and 8.

Analysis Of Structure-Medium Interaction

In Fig. 5, the structure and the medium are assumed to be in "welded contact" along their common interface and it is only in this interface that internal interaction forces are assumed to be developed to form the necessary coupling. By the use of the non-orthogonal model, any boundary configuration may be formed, and the applicability of the model is easily illustrated if a rectangular area be assumed as the working space. It is ideal to locate the limiting boundaries as far as practicable from the structure from the structure in order to minimize, if not entirely eliminate, unwanted reflections from these boundaries. On the other hand, distant limiting planes mean more lumped masses and more degrees of freedom.

In this investigation, a horizontal shear wave is generated from the bottom boundary and the mass points on that plane are constrained from moving vertically while free to move in the x-direction. The mass points on the vertical side boundaries are constrained from moving vertically and having the same horizontal motions as the corresponding adjacent mass points.

At the interaction planes, it is assumed that there is no slipping nor separation between the medium mass(es) and the corresponding structure mass, and that each set of interacting masses possesses two degrees of freedom because rotation of the masses due to the deformation of the structure is neglected.

The equations of motion of the interacting masses in the generalized coordinate direction take the form:

$$(m_m + m_s)\ddot{u}_q + (R_s + R_m) = 0 \qquad (19)$$

where the structure resistance R_s is derived from the shear in the members and the medium resistance R_m is coming from the stress point(s). As soon as the common acceleration $\ddot{u}_q$ is determined, the free-body diagram of either the structure or the medium mass will illustrate the magnitude of the interaction force component.

The recursive equations for the present plane strain problems are integrated numerically by computing the velocities and displacements of the lumped masses from known, or otherwise assumed, accelerations by the use of Newmark's β-method equations:

$$\begin{aligned} u_i^t + \Delta^t &= u_i^t + (1-\gamma)\ddot{u}_i^t\Delta t + \gamma\ddot{u}_i^{t+\Delta t}\Delta t \qquad (20)\\ &= u_i^t + \Delta u_i \\ u_i^t + \Delta^t &= u_i^t + u_i^t\Delta t + (\tfrac{1}{2} - \beta)\ddot{u}_i^t\Delta t^{-2} + \beta u_i^{t+\Delta t}\Delta t^{-2} \\ &= u_i^t + \Delta u_i \end{aligned}$$

for any mass i. It is pointed out that γ in the above Equations (20) must be 1/2 if spurious oscillations are to be avoided. β is a parameter which defines the assumed shape of the acceleration-time curve. For most structural applications, β equals 1/4 represents uniform acceleration within the time interval considered, such that the value is equal to the arithmetic average of the acceleration at the beginning and at the end of the interval; and β equals 1/6 is equivalent to a straight-line variation of assumed acceleration. The classical theory of wave propagation in solids assumes that the velocity of any point is zero until the wave reaches the point in question, at which time, the velocity jumps instantaneously to a finite value. It is therefore proposed that βequals zero may give the closest approximation with the classical theory. The method is essentially an iterative procedure, but if β is assumed zero, which is done in this study, displacements are computed directly without iteration, while velocities are integrated in a single iteration.

The numerical process also requires that the space mesh and time interval to be used in the computational scheme must meet certain mathematical requirements necessary for stable calculations. It is recommended that:

$$\Delta t \leq 0.707 \frac{\Delta x}{C_1} \qquad (21)$$

where Δt is the time interval selected, Δx is the space mesh size, and C_1 is the dilatational wave velocity defined in Equation (7).

The key step in the modal method of dynamic analysis is be done, equations of constraints must be independent, and sufficient conditions must be present to prevent rigid body movements. This will guarantee that the stiffness matrix of the structure is non-singular and the determination of the mode shapes becomes a valid eigenvalue problem. In the stiffness formulation, the generalized determinantal equation derived from the equations of motion of the undamped free vibration of the masses of the structure is:

$$[m]^{-1} [k] \{q\} - w^2 [q] = 0 \qquad (22)$$

In the above Equation (22), [m] and [k] are respectively the generalized mass matrix and generalized stiffness matrix, and w^2 is a scalar constant.

The response spectrum gives the most comprehensive information regarding maximum response in terms of dispacement, force, acceleration, etc. of a linear one-degree-of-freedom system due to a given input. Presently, the primary interest is that of ground motion, and the maximum responses may be taken a sthe maximum relative displacement between the mass and its support, the force in the spring, or the absolute acceleration of the mass. All of these quantities are recognizably interrelated, but it is convenient to plot the maximum response in terms of a quantity with the dimensions of velocity. This so-called pseudo-velocity is plotted as ordinates in logarithmic scale and the abscissa of the plot is the natural frequency (or period) of the system. If the abscissa is also plotted in logarithmic scale, the response spectra will show distinct characteristics and general shapes.

THE NUMERICAL PROCESS

The main computer program is written in the POST* languate with Fortran links. Auxiliary and support programs, including a plotting routine, are entirely in Fortran. Although there is no attempt to duplicate exactly a particular medium such as soil, the physical constants of the problem are assumed more or less in such a way as to simplify the properties of the medium. After deciding on the mesh size as required by stability and convergence requirements, the medium is divided into subregions to allow for greater economy in core space in the computer. Member and mass incidences are generated and finally, displacements, velocities, accelerations and stresses are initially set to zero, and the stress points are considered elastic. Boundary conditions are imposed on appropriate mass points.

A typical cycle of operation starts when all physical quantities are assumed or are known at a certain time. The numerical solution is carried out in incremental fashion during successive time intervals of Δt. The incremental change in the input quantity itself is interpolated for the interval, and incremental changes in the velocity and displacement are integrated. The strain-displacement relations of quation (11) give the incremental changes in the strains; relations between stresses and strains according to Equation (3a) give the incremental changes in stresses. Knowing the stresses, the resistances can be calculated. Finally, the equations of motion determine the accelerations of the masses in incremental fashion also. At the end of the cycle, all response quantities are updated such that:

Present Quantity equals Previous Quantity plus Incremental change for the present time interval

Shears, bending moments and point rotations for the structure are solved by means of back substitution formulas. All updated quantities are compared to their values before the present time interval, and in this way, maximum magnitudes are determined.