A Mixed-Binary Linear Model for Fleet Routing and Multiple Product-types Distribution

by

Cesar G. Tapia*

This paper presents a mathematical programming approach to solving the problem of routing a fleet of vehicles for the distribution of a number of product-types in a network where each node represents a client with specific demands. The proposed mathematical model utilizes binary and real valued decision variables and linear formulations of the objective function and the constraints.

The fleet routing for multiple product-types distribution problem can be seen as an extension and/or a combination of three classical operations research problems, namely:

(a) the Travelling Salesman's Problem (for fleet routing),
(b) the Knapsack Problem (for multiple product-types loading), and
(c) the Transportation Problem (for products distribution).

For the fleet routing concern, the fleet is composed of a number of vehicles such as tankers, or vans, or airplanes. The vehicles may be classified into different types according to certain attributes such as their loading capacities, cost of operations, loading costs, among others. The fleet is supposed to cover certain routes in a network of nodes representing clients with specific demands for a number of product-types. The route of a vehicle is a cyclic path composed of arcs or segments between adjacent nodes where each vehicle starts off from a specific node (initial node) and ends up at the same node (terminal node = initial node).

The fleet routing problem is therefore a variation of the travelling salesman's problem. In the latter problem, the salesman represents a single vehicle which is required to visit all the nodes in the network. In the former problem, the fleet represents one or more vehicle (salesmen), each of which is required to visit a subset of nodes which, together with the arcs connecting adjacent nodes, should constitute the cyclic route of a vehicle. The objective in

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* Professor of Mathematics at the University of the Philippines, College of Science, Department of Mathematics.

both problems is to find the route associated with the minimum total distance (cost) such that none of the nodes is ever visited more than once (excepting the initial node) by the same vehicle.

An inherent problem in the multiple product-types distribution concern is the determination of the associated loading requirement for each vehicle per segment (arc) in its route. A vehicle represents a number of (one or more)resources having a limited value (or values) which must not be exceeded. Each resource, such as the total load capacity (in terms of the maximum weight and volume capacities) of the vehicle must be utilized optimally by ensuring that the best mix of units of the product-types is loaded into a vehicle and moved through each segment of the route.

The problem of determining the optimal loading strategy for the multiple product-types per vehicle throughout its route can be seen as an extended knapsack problem. In the simplest knapsack problem, the knapsack represents a single vehicle in the fleet routing problem with only one limited resource such as its volume capacity. In both problems, there is a need to determine the best mix of units of multiple product-types to be loaded into the knapsack or vehicle, where the objective is to achieve the optimal total utility value such as the minimum total loading cost. It is clear that the fleet routing and multiple product-types distribution problem actually involves solving a knapsack-type of problem for each vehicle and for each segment in its route because it demands the determination of the optimal loading combination of units of the multiple product-types to be transported by each vehicle through each segment of its route.

For the multiple product-types distribution concern, the demands for each of the product-types in every node (or client) in the network must be satisfied. The problem of determining the optimal distribution of all the product-types can be seen as an extension of the simplest transportation problem. In the latter problem, there is only one type of product, one set of source nodes and another set of destination nodes, and the associated graph is a bi-partite graph where each link (arc) between a source node and a destination (client) node is assumed to be serviced by a vehicle. In the former problem, there are more than one product-type, and for each vehicle there is only one source node but there may be one or more destination (client) nodes. In both problems, the objective is to minimize the total cost of moving the products from the source nodes to the destination nodes whereby the demands at each destination node for the units of all the product-types are satisfied.

Definition of Symbols

Let us define the following symbols which can be utilized in the formulation of the objective function and the constraints for the fleet routing and multiple product-types distribution problem.

The following symbols represent the parameters of known values.

i,j,n = distribution points (nodes), where i,j,n = 1 represents the source (or terminal) node for a problem situation with all the vehicles having the same source node (i.e., there is only one major source (node) for all the product-types, such as a refinery for petroleum products).

k = vehicle type

l = product-type

$ds_{i,j}$ = distance from node i to node j

fcs_k = operating cost per unit distance of running vehicle k

$lcs_{k,l}$ = loading cost per unit distance per unit of load of product-type using vehicle k

$dm_{i,l}$ = demand at node i for product-type l

cp_k = loading capacity of vehicle k

The following symbols represent the decision variables.

$cost$ = total transportation cost

$x_{k,i,j,l}$ = quantity of product-type l to be transported by vehicle k from node i to node j

$xknl_{k,n,l}$ = quantity of product-type l to be disposed by vehicle k at node n

$$t_{k,i,j} = \begin{cases} 1, & \text{if vehicle } k \text{ uses arc } (i,j) \\ 0, & \text{otherwise} \end{cases}$$

A Mixed-Binary Linear Model

The objective function representing the total transportation cost in terms of the parameters and the decision variables defined above can be formulated as a linear function as follows:

$$\text{Cost} = \sum_i \sum_j \sum_k ds_{i,j} * t_{k,i,j} * fcs_k + \sum_i \sum_j \sum_k \sum_l ds_{i,j} * x_{k,i,j,l} * lcs_{k,l}$$

The first summation represents the cost components which are dependent on the distances covered by the vehicles and the fixed costs per unit distance entailed while utilizing the vehicles. The second summation represents the cost components which are dependent on the distances and the loading costs for transporting the products.

The following are the relevant constraints which all have linear formulations in terms of the parameters and the decision variables defined above.

1. Demand Constraints:

For every node n (except n=1), and for every product l:

$$\sum_{k} xknl_{k,n,l} \geq dm_{n,l}$$

The sum of the quantities of product-type l disposed at node n by all the vehicles must be at least equal to the demand for the product-type l.

2. Supply Constraints:.

For node i = 1, and for every product l:

$$\sum_{j \neq 1} \sum_{k} x_{k,i,j,l} \geq \sum_{j \neq 1} dm_{j,l}$$

From the source node (say, a single refinery at node i = 1), the quantities of product l to be transported by the vehicles to the other nodes should in total be at least equal to the total of the demands at all the nodes.

3. Destination Constraints:

For every vehicle k, and for every node i:

$$\sum_{j \neq i} t_{k,i,j} \leq 1$$

From a node i, a vehicle can have at most one destination node j.

4. Exit Constraints:

For every vehicle k, and for every node $i \neq 1$:

$$\sum_{n \neq i} t_{k,n,i} = \sum_{j \neq i} t_{k,i,j}$$

If a vehicle k does not pass through node i (i.e., the left-side summation = 0), then it should not exit from node i (i.e., the right-side summation = 0), and if it does (i.e., the left-side summation = 1), then it must exit from node i (i.e., the right-side summation = 1).

5. Capacity Constraints:

For i = 1, and for every vehicle k:

$$\sum_{j \neq 1} \sum_{l} x_{k,i,j,l} \leq cp_k$$

The total of the quantities of the product-types transported by a vehicle from the source node should not exceed the capacity of the vehicle.

6. Flow Conservation Constraints:

For every vehicle k, for every node $n \neq 1$, and for every product l:

$$\sum_{i \neq n} x_{k,i,n,l} - xknl_{k,n,l} = \sum_{j \neq n} x_{k,n,j,l}$$

Except for the source node, the quantity of a product transported by a vehicle to a node n minus the quantity disposed at that node by the vehicle must be equal to the quantity transported from node n to the next node.

7. No Loop Constraints:

For every node i (or j), and for every vehicle k:

$$t_{k,i,j} = 0 \text{ if } i = j$$

8. Transported Quantity Constraints:

For every vehicle k, for every node i (≠j), and for every product l:

$$x_{k,i,j,l} \leq t_{k,i,j} * M$$

where M is a sufficiently large number (e.g., $M = \sum_{k} cp_k$)

Variables $x_{k,i,j,l} = 0$ if route segment (i,j) is not used by vehicle k (i.e., $t_{k,i,j} = 0$), otherwise it should not exceed the total tonnage capacity of vehicle k.

9. Zero Load Constraints:

For every vehicle k, for every node $i \neq 1$, for every product l, and j = 1:

$$x_{k,i,j,l} = 0$$

Each vehicle must have a zero load when it returns to the source (or terminal) node.

Sample Computational Results

To demonstrate the applicability of the mathematical formulations above, the following sample data are used for a fleet routing and multiple products distribution problem and a GAMS program (given in the Appendix) provide the results described below.

1. There are six nodes representing i and j:
   REFINERY, DEPOT1, DEPOT2, DEPOT3, DEPOT4, and DEPOT5

2. There are two vehicles representing tankers, k, of two types
   TYPE-1 and TYPE-2

3. There are two petroleum products, l:
   PROD1 and PROD2

4. The fixed cost fcs(k) per km of running tanker k are:
   fcs(TYPE-1) = 10 and fcs(TYPE-2) = 6

5. The tonnage capacity cp(k) of tanker k are:
   cp(TYPE-1) = 500 and cp(TYPE-2) = 800

6. The following table gives the loading costs, $lcs_{k,l}$ per ton per km of running tanker k for product l:

$lcs_{k,l}$	PROD 1	PROD2
TYPE-1	10	6
TYPE-2	4	8

7. The following table gives the distances, $ds_{i,j}$ from distribution point i to point j:

$ds_{i,j}$	REFINERY	DEPOT1	DEPOT2	DEPOT3	DEPOT4	DEPOT5
REFINERY	-	5	11	10	8	7
DEPOT1	5	-	6	8	8	8
DEPOT2	11	6	-	4	7	9
DEPOT3	10	8	4	-	6	10
DEPOT4	8	8	7	6	-	6
DEPOT5	7	8	9	10	6	-

8. The following table gives the demands, $dm_{i,l}$ at depot i for petroleum product l

$dm_{i,l}$	PROD1	PROD2
REFINERY	0	0
DEPOT1	100	100
DEPOT2	200	100
DEPOT3	200	70
DEPOT4	300	30
DEPOT5	50	20
TOTAL	850	320

For the above data set used in the mathematical programming model for this problem, GAMS generates 209 single equations, 213 single real variables, and 62 discrete variables. The following optimal route is obtained.

For tanker TYPE-1:

REFINERY → DEPOT1 → DEPOT2 → DEPOT3 → DEPOT5 →REFINERY

For tanker TYPE-2:

REFINERY → DEPOT4 → DEPOT3 → DEPOT2 → DEPOT5 →REFINERY

The optimal initial loads of each tanker originating at the REFINERY are:

	PROD1	PROD2	Maximum Capacity
TYPE-1	100	290	500
TYPE-2	750	30	800
Total Demand	850	320	

The optimal quantity of product l moved from point i to j by tanker k:

For tanker TYPE-1:

	PROD1	PROD2
REFINERY - DEPOT1	100	290
DEPOT1-DEPOT2	0	190
DEPOT2-DEPOT3	0	90
DEPOT3-DEPOT5	0	20
DEPOT5-REFINERY	0	0

For tanker TYPE-2:

	PROD1	PROD2
REFINERY - DEPOT4	750	30
DEPOT4-DEPOT3	450	0
DEPOT3-DEPOT2	250	0
DEPOT2-DEPOT5	50	0
DEPOT5-REFINERY	0	0

The optimal quantity of product l disposed by tanker k at node n are:

For tanker TYPE-1:

	PROD1	PROD2
DEPOT1	100	100
DEPOT2	0	100
DEPOT3	0	70
DEPOT5	0	20

For tanker TYPE-2:

	PROD1	PROD2
DEPOT4	300	30
DEPOT3	200	0
DEPOT2	200	0
DEPOT5	50	0

The optimal total fixed cost = 524, while the optimal total loading cost = 66420, which together make up of the optimal total transportation cost = 66944.

Possible extensions of the mathematical programming formulation:

The following additional requirements in the fleet routing and multiple products distribution problems can be incorporated in the proposed mathematical model.

a. Changes in demands (either deterministic or probabilistic)
b. Closing down a depot or putting up additional depot(s)
c. Decreasing / increasing the number of vehicles
d. Decreasing / increasing the number of product-types
e. Determining optimal capacities of the vehicles
f. Increasing the number of objective functions (MOMP)
e.g., Separate fixed cost and loading cost with additional opportunity cost when initial supply of products is limited or supply is short of demand.

References

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4. Rutherford, T., 1994. *The GAMS/MPSGE and GAMS/MILES User Notes.*

Appendix

A GAMS Program for the Fleet Routing and Multiple Products Distribution Problem

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SETS   I distribution points
       /REFINERY,DEPOT1,DEPOT2,DEPOT3,DEPOT4,DEPOT5/
       K tankers   / TYPE-1, TYPE-2 /
       L petroleum products / PROD1, PROD2 /
ALIAS (I,J,N);
PARAMETERS FCS(K)  fixed cost per km of running tanker k
       /TYPE-1  10, TYPE-2  6/
  CP(K)  tonnage capacity of tanker k
       / TYPE-1 500 , TYPE-2 800/
PARAMETER M multiplier to define an upper bound of x due to routing
       M = SUM(K,CP(K));
TABLE LCS(K,L) loading cost per ton per km of running tanker k for product l
                    PROD1       PROD2
       TYPE-1        10           6
       TYPE-2         4           8

TABLE DS(I,J)   distances from distribution point i to point j
             REFINERY  DEPOT1   DEPOT2   DEPOT3   DEPOT4  DEPOT5
REFINERY     1000000      5        11       10        8       7
 DEPOT1          5    1000000       6        8        8       8
 DEPOT2         11        6    1000000       4        7       9
 DEPOT3         10        8         4   1000000       6      10
 DEPOT4          8        8         7        6  1000000       6
 DEPOT5          7        8         9       10        6  100000

TABLE DM(I,L)   demand at depot i for petroleum product l
              PROD1   PROD2
    REFINERY     0       0
    DEPOT1     100     100
    DEPOT2     200     100
    DEPOT3     200      70
    DEPOT4     300      30
    DEPOT5      50      20

VARIABLES  X(K,I,J,L)    tons of product l moved by tanker k from node i to j
           XKNL(K,N,L)   tons of prod l disposed by tanker k at node n
           T(K,I,J)      binary variable: 1 means tanker k passes pt. i to j,
                              0 otherwise
           COST          total transportation cost;
POSITIVE VARIABLE X,XKNL;
BINARY VARIABLES T;
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EQUATIONS
        COSTEQN            total cost equation
        DEMAND(N,L)        demand constraint for point n for product l
        SUPPLY(I,L)        supply constraint from the refinery for prod 1
        DESTIN(K,I)        unique destination constraint
        EXIT(K,I)          exit rqmt for tanker k visiting depot i
        CAPACITY(K)        capacity constraint for tanker k
        FLOW(K,N,L)        flow conservation at node n for product l, tanker k
        TZERO(K,I,J)       no loop constraints for i = j
        XBOUND(K,I,J,L)    boundary constraint on x due to routing rqmt
        ZEROXRET(I,J,K,L)  zero load on return to the refinery for j = 1;

COSTEQN..  COST =E= SUM((I,J,K,L)$(ORD(I) NE ORD(J)), X(K,I,J,L)*DS(I,J)
           * LCS(K,L)) + SUM((K,I,J)$(ORD(I) NE ORD(J)), DS(I,J)*
           T(K,I,J)*FCS(K));
DEMAND(N,L)$(ORD(N) NE 1).. SUM(K, XKNL(K,N,L) ) =E= DM(N,L);
SUPPLY(I,L)$(ORD(I) EQ 1).. SUM((J,K)$(ORD(I) NE ORD(J)),X(K,I,J,L))
                =G= SUM(J$(ORD(J) NE 1),DM(J,L));
DESTIN(K,I)..  SUM(J$(ORD(J) NE ORD(I)), T(K,I,J)) =L= 1;
EXIT(K,I)$(ORD(I) NE 1).. SUM(J$(ORD(J) NE ORD(I)), T(K,I,J)) =E=
                SUM(N$(ORD(N) NE ORD(I)), T(K,N,I)) ;
CAPACITY(K)..  SUM((J,L)$(ORD(J) GT 1), X(K,"REFINERY",J,L)) =L= CP(K);
FLOW(K,N,L)$(ORD(N) GT 1).. SUM(I$(ORD(I) NE ORD(N)), X(K,I,N,L)) -
                XKNL(K,N,L) =E= SUM(J$(ORD(J) NE ORD(N)), X(K,N,J,L));
TZERO(K,I,J)$(ORD(J) EQ ORD(I)).. T(K,I,J) =E= 0;
XBOUND(K,I,J,L)$(ORD(I) NE ORD(J)).. X(K,I,J,L) =L= T(K,I,J)*M;
ZEROXRET(I,J,K,L)$((ORD(J) EQ 1)$(ORD(I) NE 1)).. X(K,I,J,L) =E= 0;
MODEL FLEET / ALL /;
SOLVE FLEET MINIMIZING COST USING MIP;
PARAMETER
       LOAD(K,L) optimal initial load of tanker k for product l;
       LOAD(K,L) = SUM(J,X.L(K,"REFINERY",J,L));
PARAMETERS
 OPTFCOST = optimal total fixed cost
 OPTLCOST = optimal load cost;
 OPTFCOST = SUM((K,I,J), DS(I,J)*T.L(K,I,J)*FCS(K));
 OPTLCOST = SUM((I,J,K,L), X.L(K,I,J,L)*DS(I,J)*LCS(K,L));
DISPLAY LOAD, X.L, XKNL.L, T.L, OPTFCOST, OPTLCOST, COST.L;
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