Fission and fusion schemes and the action of the diagonal group $D(T, n)$

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ABSTRACT

This paper is a closer look at associate classes of an association scheme. The orbitals of a subgroup $D(T, n)$ of the full automorphism group of the association schemes served as building blocks for the fusion schemes.

Keywords: association scheme, fusion and fission schemes, orbital, intersection number, permutation, group action

SOME IMPORTANT DEFINITIONS

Definition 1. *Let Ω be any non-empty set. Let R be a partition of $\Omega \times \Omega$. (Ω, R) is a* **coherent configuration** *if the following axioms are satisfied:*

(a) The set $\{(\alpha, \alpha) \mid \alpha \in \Omega\}$ is a union of some elements of R. This set is called the diagonal.

(b) For each class $C \in R$, the transpose C^* which is defined as $\{(\beta, \alpha) \mid (\alpha, \beta) \in C\}$ is a class in R.

(c) For all C_i, C_j and C_k in R and for any $(\alpha, \beta) \in C_k$, there is a non-negative number p^k_{ij} such that the number of γ satisfying

$$(\alpha, \gamma) \in C_i \wedge (\gamma, \beta) \in C_j \qquad (1)$$

is equal to p^k_{ij}. This number is independent of the choice of $(\alpha, \beta) \in C_k$.

If G is a permutation group on Ω and R is the set of orbitals then, (Ω, R) is called the *coherent configuration associated with G*. The group G is said to be *generously transitive* on Ω if given any $(\alpha, \beta) \in \Omega \times \Omega$, there is an element $g \in G$ such that

$$(\alpha, \beta)^g = (\beta, \alpha) \qquad (2)$$

Replace item (b) by

(b') For each class C, $C^* = C$.

With (b'), the coherent configuration is called *association scheme*. If item (a) is replaced by

(a') The diagonal is one class in R,

then, (Ω, R) with R as a set of orbitals is called an *association scheme associated with the transitive permutation group G*. A group G with association scheme (Ω, R) which is generously transitive is necessarily transitive on the elements of Ω.

Some relations between association schemes are defined here.

Definition 2. *Consider the following association schemes.*

$$A1 = (\Omega, R_1) \wedge A_2 = (\Omega, R_2) \qquad (3)$$

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The association scheme A_1 is finer than A_2 if for all $C_k \in R_1$, there exists $C_l \in R_2$ such that $C_k \subset C_l$. Also, A_2 is said to be coarser than A_1. The association scheme A_1 is fission scheme while A_2 is called a fusion scheme.

THE DIAGONAL GROUP

Diagonal groups were defined by Cameron (Cameron, 1999) and the following are their definition in terms of generators (Alejandro, et al., 2003). A diagonal group $D(T, n)$ where T is a group and n a positive integer is a permutation group acting on a non-empty set Ω given by

$$\Omega = \{[x_1, x_2, \ldots, x_n] | x_i \in T\} \qquad (4)$$

The following are the generators of $D(T, n)$:

(a) the group T^n acting by right translation, that is the permutations

$$[x_1, x_2, \ldots, x_n] \mapsto [x_1 t_1, x_2 t_2, \ldots, x_n t_n] \qquad (5)$$

for $t_1, t_2, \ldots, t_n \in T$;

(b) the automorphism group of T, acting coordinate-wise, that is,

$$[x_1, x_2, \ldots, x_n] \mapsto [x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha] \qquad (6)$$

for $\alpha \in Aut(T)$;

(c) the symmetric group S_n, acting by permuting the coordinates, that is,

$$[x_1, x_2, \ldots, x_n] \mapsto [x_{1\pi}, x_{2\pi}, \ldots, x_{n\pi}] \qquad (7)$$

where $\pi \in S_n$;

(d) the permutation τ acting as follows:

$$[x_1, x_2, \ldots, x_n] \mapsto [x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1}x_n]. \qquad (8)$$

Another way of looking at diagonal groups is by their action on the right cosets of a point-stabilizer. Define a group G as follows:

$$G = T^{n+1}(Out(T) \times S_{n+1}) \qquad (9)$$

Consider the subgroup

$$H = Aut(T) \times S_{n+1} \qquad (10)$$

The inner automorphisms are identified with the diagonal subgroup

$$\{(t, t, \ldots, t) \mid t \in T\} \qquad (11)$$

of T^{n+1}. This is how diagonal groups are described in the O'Nan-Scott Theorem. Each right coset of H in G has a unique coset representative $(1, x_1, \ldots, x_n) \in T^{n+1}$. This representative is denoted $[x_1, \ldots, x_n]$. Now, let us observe the action of G on the coset representatives:

$$\begin{aligned}
[x_1, \ldots, x_n](1, t_1, \ldots, t_n) &= H(1, x_1, \ldots, x_n)(1, t_1, \ldots, t_n) \\
&= H(1, x_1 t, 1 \ldots, x_n t_n) \\
&= [x_1 t_1, \ldots, x_n t_n].
\end{aligned} \qquad (12)$$

Thus we obtain the permutations in (a). A conjugation by an element $t \in T$ is uniquely a product of an element of T^{n+1} with first coordinate 1 and the diagonal element $(t, \ldots, t)$ as seen below:

$$\begin{aligned}
[x_1, \ldots, x_n](t, \ldots, t) &= H(1, x_1, \ldots, x_n)(t, \ldots, t) \\
&= H(t, x_1 t, \ldots, x_n t) \\
&= H(1, t^{-1}x_1 t, \ldots, t^{-1}x_n t) \\
&= [t^{-1}x_1 t, \ldots, t^{-1}x_n t]
\end{aligned} \qquad (13)$$

Hence, $Inn(T)$ is a subset of T^{n+1}. The remaining automorphisms in (b) are exactly the outer automorphisms of T. Observe the following action of τ:

$$\begin{aligned}
[x_1, \ldots, x_n]^\tau &= H(1, x_1, \ldots, x_n)^\tau \\
&= H(x_1, 1, x_2, \ldots, x_n) \\
&= H(1, x_1^{-1}, x_2, \ldots, x_1^{-1}x_n) \\
&= [x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1}x_n]
\end{aligned} \qquad (14)$$

The symmetric group S_{n+1} is generated by the symmetric group S_n in (c) and the transposition $\tau = (1, 2)$ in (d).

Diagonal groups $D(T, n)$ where $n \geq 8$ and T abelian are generously transitive permutation groups. All orbitals in $Orbl(\Omega, D(T, n))$ are symmetric and this gives a commutative association scheme. Some association schemes result from fission schemes. For the case of $Orbl(\Omega, D(T, n))$, a fission scheme is formed by looking

at a proper subgroup K of $D(T, n)$. A special set of generators is used to construct the subgroup K. By taking union of elements of $Orbl(\Omega, K)$, the association scheme from $Orbl(\Omega, D(T, n))$, is obtained. The character tables corresponding to the basis algebra of $Orbl(\Omega, K)$ and $Orbl(\Omega, D(T, n))$ will be investigated by making each entry a function of T and n.

THE ORBITALS

Let T be a permutation group acting on itself by right translation. Let $AutT$ be the automorphism group of T acting on the elements of T naturally. The semi-direct product $T^2 \times OutT$ acts on T the following way:

$$x^{((t_1, t_2), \gamma)} = (t_1^{-1} x t_2)^\gamma \qquad (15)$$

for all $x \in T$ and for all $(t_1, t_2)^\gamma \in T^2 \times AutT$.

The *orbitals* of T^2 acting on T is the set of orbits resulting from the induced action of T^2 on $T \times T$. The orbitals of $OutT$ in T is defined similarly. The two actions are shown below:

$$(x_1, x_2)^{(t_1, t_2)} = (t_1^{-1} x_1 t_2, t_1^{-1} x_2 \cdot t_2)$$
$$(x_1, x_2)^\gamma = (x^\gamma_1, x^\gamma_2) \qquad (16)$$

The discussion below shows that the orbitals of $T^2 \times OutT$ can be computed using the above orbitals.

The following is a detailed discussion of an important part of the proposition. Let (α_1, β_1) be any element of $(\Omega \times \Omega)$. Suppose that for some $\gamma \in AutT$, $(\alpha_1, \beta_1)^\gamma = (\overline{\alpha}, \overline{\beta})$. Also, for some $(t_1, t_2) \in T \times T$ $(\overline{\alpha}, \overline{\beta})^{(\overline{t_1}, t_2)} = (\alpha_2, \beta_2)$. Then, (α_2, β_2) is in a T^2-orbital containing an image of (α_1, β_1) under an automorphism γ.

Proposition 1. *Let T be a group and $OutT$ its outer automorphism group. Then, an orbital O of $T^2 \times OutT$ containing the element (α_1, β_1) in $\Omega \times \Omega$ in the action previously defined is the union of all T^2-orbitals and all $AutT$-orbitals containing images of (α_1, β_1).*

Proof. Let O be an orbital of $T^2 \times AutT$ containing (α_1, β_1). Suppose that (α_2, β_2) is an element of O satisfying the following:

(a) For $\iota \in AutT$
$$(\alpha_1, \beta_1)^{((t_1, t_2), \iota)} = (t_1^{-1} \alpha_1 t_2, t_1^{-1} \beta_1 t_2)$$
$$= (\alpha_2, \beta_2) \qquad (17)$$

(b) For $t \in T$
$$(\alpha_1, \beta_1)^{((1,1), \gamma)} = (\alpha_1^\gamma, \beta_1^\gamma)$$
$$= (\alpha_2, \beta_2) \qquad (18)$$

(c) For some non-identity elements t_1, $t_2 \in T$ and also non-identity auto-morphism $\gamma \in AutT$

$$(\alpha_1, \beta_1)^{((t_1, t_2), \gamma)} = ((t_1^{-1} \alpha_1 t_2)^\gamma, (t_1^{-1} \beta_1 t_2)^\gamma)$$
$$= (\alpha_2, \beta_2) \qquad (19)$$

In case (a) (α_2, β_2) is trivially in the T^2-orbital containing (α_1, β_1). In case (b) (α_2, β_2) is again trivially in the $AutT$-orbital containing (α_1, β_1). For case (c), let $\overline{\alpha} : = \alpha_1^\gamma$, $\overline{\beta} : = \beta_1^\gamma$, $\overline{t_1} = t_1^\gamma$ and $\overline{t_2} : = t_2^\gamma$. Then

$$(\alpha_2, \beta_2) = (\overline{t_1}^{-1} \overline{\alpha} \overline{t_2}, \overline{t_1}^{-1} \overline{\beta} t) \qquad (20)$$

This proves that (α_2, β_2) is in the T^2-orbital containing $(\overline{\alpha}, \overline{\beta})$ which in turn is the image of (α_1, β_1) under γ.

The three cases cover all possible images of (α_1, β_1) under the induced action of the semi-direct product $T^2 \times AutT$.

Therefore,

$$\left(\bigcup O_T\right) \bigcup \left(\bigcup OAutT\right) \subset O \qquad (21)$$

where the union is taken over all orbitals containing images of (α_1, β_1). $\qquad \square$

As a remark, to get the orbital O containing (α_1, β_1) under the semi-direct product $T^2 \times AutT$, all $AutT$-orbitals containing all elements of the T^2-orbital must be listed. Additionally, all T^2-orbitals containing all elements of the $AutT$-orbital containing (α_1, β_1) must also be listed. This result was used in a GAP program (The GAP Group, GAP- Groups, Algorithms and Programming, Version 4, http://www.gap-system.org) implemented to compute fission and fusion schemes.

THE SET OF ORBITALS AS THE FINEST ASSOCIATION SCHEME

The fusion schemes will be constructed based on the fact that the finest association scheme preserved by

$T^{n+1} \times OutT$ is its set of orbitals resulting from its action on T.

Recall that an automorphism of an association scheme is a permutation of the set of points Ω which preserves associate class C_i for each $i = 0, 1, ..., m$ where m is the number of associate classes.

The set of all orbitals O forms association scheme as discussed here. Consider the orbitals O_i, O_j and O_k which are not necessarily distinct. Let $(\alpha, \beta) \in O_k$. It can be shown that the intersection number p^k_{ij} is independent of the choice of (α, β). Denote by G the group $T^{n+1} \times OutT$. Suppose that p^k_{ij} is the non-negative number which gives the cardinality of the set

$$\{\gamma \,|\, (\alpha, \gamma) \in O_i \wedge (\gamma, \beta) \in O_j\} \tag{22}$$

Let $(\delta, \lambda) \in O_k$. It can be shown that (δ, λ) has the same intersection number. Let $g \in G$ such that $(\delta, \lambda) = (\alpha, \beta)^g$. Then $\{\gamma^g \,|\, (\alpha^g, \gamma^g) \in O_i \wedge (\gamma^g, \beta) \in O_j\} = \{\gamma^g \,|\, (\delta, \gamma^g) \in O_i \wedge (\gamma^g, \lambda) \in O_j\}$. Since g is a permutation of T, the cardinality of this set is constant given by p^k_{ij}. This means that the intersection number is independent of the choice of (α, β).

The set of orbitals O is the finest association scheme preserved by the the permutation group G. The following is an explanation. Suppose that there is a finer association scheme preserved by G. Let $\tilde{O}_i$ be an associate class of a finer one which is properly contained in orbital O_i. For some $g \in G$, $\tilde{O}_i^g \neq \tilde{O}_i$ because the group G is transitive on the elements of its orbitals. Therefore, $\tilde{O}_i$ is not preserved by G. This contradicts our assumption that the finer association scheme is preserved by the group G.

The associate classes of fusion schemes are union of orbitals of $T^{n+1} \times OutT$ acting on the elements of T. The group $T^{n+1} \times OutT$ preserves these associate classes because it preserves its orbitals. Hence, it is a subgroup of the full automorphism group of the fusion scheme.

INTERSECTION NUMBERS AND INCIDENCE MATRICES

The last axiom of the definition of association scheme deals with intersection numbers. This section discusses intersection numbers and its relation to the product of incidence matrices.

The incidence matrix A_i corresponding to the associate class C_i will be constructed by using the elements of Ω as the row and column indices. The matrix A_i is defined as follows:

$$A_i(\alpha, \beta) = \begin{cases} 1 \ if \ (\alpha, \beta) \in C_i \\ 0 \ otherwise \end{cases} \tag{23}$$

If A_i, A_j and A_k are respectively the incidence matrices of associate classes C_i, C_j and C_k, then the product $A_i \cdot Aj$ can be written as

$$\begin{aligned} A_i \cdot A_j &= p^0_{ij} \cdot A_0 + p^1_{ij} \cdot A_1 + \cdots + p^m_{ij} \cdot A_m \\ &= \Sigma_{k=1}^{k=m} p^k_{ij} A_k \end{aligned} \tag{24}$$

where m is the number of associate classes. The (α, β)-entry of the product $A_i \cdot A_j$ is non-zero if and only if there is at least one γ such that (α, γ)-entry of A_i and (γ, β)-entry of A_j are both 1. This is equivalent in saying that there is at least one γ such that $(\alpha, \gamma) \in C_i$ and $(\gamma, \beta) \in C_j$. The intersection number p^k_{ij} gives the number of such γ if and only if $(\alpha, \beta) \in C_k$. Hence, the intersection number p^k_{ij} is the coefficient of A_k in the linear expansion of $A_i \cdot A_j$.

FUSION SCHEMES FROM GROUPS ISO-MORPHIC TO Z_p

Proposition 2. *The semi-direct product $T^2 \times AutT$ is sharply 2-transitive on Ω where $T \cong Z_p$ for some prime number p.*

Proof. Let ρ be a generator of the group Z_p. Consider the following arbitrary pairs of distinct elements of $Z_p \times Z_p$.

$$(\rho^a, \rho^b) \wedge (\rho^c, \rho^d) \tag{25}$$

where $a, b, c, d \in \{0, 1, \dots, p - 1\}$. Since we would like to prove sharp 2-transitivity, we shall consider ordered pairs whose first component is distinct from the second component. Hence, the following conditions are satisfied.

$$a \not\equiv b \bmod p \text{ and } c \not\equiv d \bmod p \qquad (26)$$

Consequently,

$$a - b \not\equiv 0 \bmod p \text{ and } c - d \not\equiv 0 \bmod p \qquad (27)$$

We shall find an element $((t_1, t_2), \gamma) \in T^2 \times AutT$ such that

$$(\rho^a, \rho^b)^{((t_1, t_2),\gamma)} = (\rho^c, \rho^d) \qquad (28)$$

There is an element $(\rho^{v1}, \rho^{v2}) \in T \times T$ such that

$$(\rho^a, \rho^b)^{((\rho^{v_1}, \rho^{v_2}),\iota)} = (\rho^{-v1+a+v2}, \rho^0)$$
$$= (\rho^w, \rho^0) \qquad (29)$$

From above, we deduce the following equation:

$$b \equiv (v_1 - v_2) \bmod p \qquad (30)$$

The element w of Z_p satisfy the following:

$$w \equiv (a - b) \bmod p \qquad (31)$$

There is an ordered pair (d_1, d_2) where $d_1, d_2 \in \{0, 1, \dots, p- 1\}$ such that

$$d \equiv (-d_1 + d_2) \bmod p \qquad (32)$$

Now, let $x \in \{0, 1, \dots, p - 1\}$ such that

$$x \equiv (c - d) \bmod p \qquad (33)$$

Then, using the assumptions above, we have

$$w \not\equiv 0 \bmod p \text{ and } x \not\equiv 0 \bmod p \qquad (34)$$

It is known that the full automorphism group $AutT$ of T is transitive on its non-identity elements. Hence, there exists $\gamma \in AutT$ such that $(\rho^w)^\gamma = \rho^x$. Thus,

$$(\rho^a, \rho^b)^{((\rho^{v_1}, \rho^{v_2}),\iota), ((\rho^0, \rho^0),\gamma)} = (\rho^x, \rho^0) \qquad (35)$$

Applying $((\rho^{d_1}, \rho^{d_2}),\iota)$, we get

$$(\rho^x, \rho^0)^{((\rho^{d_1}, \rho^{d_2}),\iota)} = (\rho^{x-d_1+d_2}, \rho^{-d_1+d_2})$$
$$= (\rho^c, \rho^d) \qquad (36)$$

Therefore, the element $((t_1, t_2),\gamma)$ will be the product of the three elements of $T^2 \times AutT$ given below:

$$((\rho^{(v1}, \rho^{v2)},\iota)((\rho^0, \rho^0), \gamma)((\rho^{-d_1+d_2}, \rho^{-d_1+d_2}),\iota) \qquad (37)$$

Hence, the semi-direct product $T^2 \times AutT$ is sharply 2-transitive on T. $\square$

Corollary 1. *If $T \cong Z_p$ where p is a prime number, then the semi-direct product $T^2 \times AutT$ is generously transitive in its action on T.*

Proof. Take the two ordered pairs of distinct elements in the proposition above to be (ρ^a, ρ^b) and (ρ^b, ρ^a).$\square$

These results explain why there is only one fusion scheme for the case when the group T is isomorphic to the cyclic group Z_p of prime order. Such fusion schemes are the set of orbitals $\{C_0, C_1\}$ where C_0 is the diagonal orbital and C_1 is its complement in $\Omega \times \Omega$.

It can also be observed that for all the T considered, the orbitals are all symmetric. This implies that the semi-direct product $T^2 \times AutT$ for these groups T are generously transitive in its action on the elements of T. The following is a discussion of commutativity of incidence matrices of these orbitals. Let A_i and A_j be arbitrary incidence matrices. Recall the following equation

$$A_i \cdot A_j = p^0_{ij} \cdot A_0 + p^1_{ij} \cdot A_1 + \dots + p^m_{ij} \cdot A_m$$
$$= \Sigma_{k=1}^{k=m} p^k_{ij} A_k \qquad (38)$$

where m is the number of associate classes. The (α, β)-entry of the product $A_i \cdot A_j$ depends on the associate class where (α, β) belongs. It is p^k_{ij} if $(\alpha, \beta) \in C_k$. Since C_k is symmetric and p^k_{ij} is independent of the choice of (α, β) by axiom (c) of the definition, the (β, α)-entry of the product is also p^k_{ij}. This means that the product of incidence matrices A_i and A_j for $i, j \in \{0, 1, \dots, m\}$ is symmetric. This implies that the algebra of

matrices over the set of reals $\Re$ generated by A_0, A_1, $\cdots$, A_m is a commutative algebra.

THE DIAGONAL GROUP AND THE FUSION SCHEMES

The diagonal group $D(T, n)$ contains $T^{n+1} \times OutT$ as a proper subgroup for $n \geq 1$. If an association scheme is preserved by $D(T, n)$, then it is also preserved by every subgroup of $D(T, n)$. As a consequence, the fusion schemes preserved by $T^{n+1} \times OutT$ are candidate association schemes preserved by $D(T, n)$. A complete list of all possible association schemes preserved by $D(T, n)$ can be derived from the fusion schemes preserved by $T^{n+1} \times OutT$. This claim is shown in the following result:

Corollary 2. *All association schemes preserved by $D(T, n)$ have associate classes which are union of orbitals of $T^{n+1} \times OutT$*

Proof. Let C be an associate class in an association scheme preserved by $D(T, n)$. Suppose that C is not a union of orbitals of $T^{n+1} \times OutT$. Then there exists an orbital Oi of $T^{n+1} \times OutT$ and $O_{ic} \subset O_i$ such that $O_{ic} = C \cap O_i$. Consider $x \in O_{ic}$ and $y \in O_i - O_{ic}$. There exists $g \in T^{n+1} \times OutT$ such that $x^g = y$ and hence $C^g \neq C$. This is a contradiction because C is an associate class for $D(T, n)$. $\square$

Association schemes preserved by $D(T, n)$ are coarser as compared with the association schemes preserved by $T^{n+1} \times OutT$.

For the case when $n = 1$ and $T \cong Zp$ for some prime number p, only the trivial association scheme is preserved by $D(T, n)$. The generous transitivity of $T^2 \times OutT$ on Ω implies that of the diagonal group.

DISCUSSION

The automorphism groups of finitely generated groups can further be studied to characterize the resulting orbitals in the action of $T^2 \times OutT$. A diagonal group $D(T, n)$ is isomorphic to $(T^{n+1} \times OutT)$ o S_{n+1}. The generators are as enumerated in the introductory section of diagonal groups. The GAP algorithm may further be implemented for those cases when $n \geq 2$.

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