Study of the Perturbation to a Bose-Einstein Gas

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ABSTRACT

We developed a new approach to the perturbation theory for the effective Hamiltonian of condensate particles in Fock space. Using this new theory, we can easily analyze the effect of including a somewhat problematic term in the work of Ezawa et al. We thus showed that indeed, the inclusion of this term in the perturbation potential is justified.

INTRODUCTION

The phenomenon of the Bose-Einstein condensation, first observed (Anderson et al., 1995) for ^{57}Rb at 170 K, followed by the cases (Davis et al., 1996; Bradley et al., 1997) of ^{23}Na, ^{7}Li, and ^{1}H, has excited experimental and theoretical interests on different aspects of this quantum effect. In particular, Ezawa et al. (1998) studied the fluctuation of the condensate by modifying the Bogoliubov prescription (Bogoliubov, 1947) in replacing a_0 by $\sqrt{N_0}$ with (Ezawa & Luban, 1967; Ezawa, 1965)

$$a_0 \rightarrow \sqrt{N_0} + a_0^{'} \tag{1}$$

where N_0 is the number of condensate particles and a_0, the annihilation operator. This work was done with the modified Oppenheimer approach to perturbation theory in Fock space to obtain the effective Hamiltonian for the condensate, using $\lambda = \left(\sqrt{N_0}\right)^{-1}$ as the strength parameter. It was then shown that the fluctuations are much less than N_0, thus justifying Bogoliubov's prescription.

In this work, a term

$$H_{BC}^{(0)} = \sum_{n \neq 0} J_{no}\left(a_n^{\dagger} + a_n\right)\left(a_n^{\dagger} + a_0\right) \tag{2}$$

which is of zero-th order in λ was included in the perturbation, and the perturbation was carried out to second order in λ This is unusual in perturbation work. We shall therefore consider the contributions of these terms to higher orders to seek justification for this work.

THE SYSTEM HAMILTONIAN

The Hamiltonian for a Bose-Einstein gas in a trap is

$$H := \int \phi_A^{\dagger}(x)\left\{-\frac{\mathrm{h}^2}{2M}\Delta + \nu(x) - \mu\right\}\phi_A(x)\, d^3x + \frac{1}{2}\int \phi_A^{\dagger}(x)\phi_A^{\dagger}(x')V(x-x')\phi_A(x)\phi_A(x')\, d^3x d^3x' \tag{3}$$

where $\nu(x)$ is the trap potential and $V(x) = V(-x)$, the interaction. The former, which varies much more slowly than the latter, is the chemical potential. In terms of the new field a_0', the field operator takes the form

$$\phi_A(x) = \sqrt{N_0}\, u(x) + \phi(x) \tag{4}$$

where $\phi(x) = \sum_n a_n u_n(x)$. The operator a_0 shall henceforth be taken to mean a_0'.

Terms linear in (x) will arise in H and can be eliminated by adding and subtracting the Hartree potential

$$v_H(x) := N_0 \int V(x-x')\, u_0^2(x')\, d^3x \tag{5}$$

so that

$$\begin{aligned} H &= \int \phi_A^\dagger(x) \left\{ -\frac{\hbar^2}{2M} + v_H(x) + v(x) - \mu \right\} \phi_A(x)\, d^3x \\ &+ \frac{1}{2} \int \phi_A^\dagger(x) \phi_A^\dagger(x') V(x-x') \phi_A(x) \phi_A(x')\, d^3x d^3x' \\ &- \int \phi_A^\dagger(x) v_H(x) \phi_A(x)\, d^3x \end{aligned} \tag{6}$$

The functions $u_n(x)$ are chosen to be real eigenfunctions of

$$h = -\frac{\hbar^2}{2M}\Delta + v(x) + v_H(x) \ . \tag{7}$$

In view of the short range of the interaction as compared to the wavelength of the atoms, people take the delta-function approximation

$$v(x-x') = g\delta(x-x') \quad \text{and} \quad g = \frac{4\pi\hbar^2 a}{M} \tag{8}$$

where a is the scattering length of the atoms, so that Eq. (7) simplifies as

$$h = -\frac{\hbar^2}{2M}\Delta + v(x) + gN_0 u_0^2(x) \tag{9}$$

Although we have a set of non-linear eigenvalue equations, it is easy to reflect that we still have a complete orthonormal set of eigenfunctions.

The Hamiltonian of the system can now be written as

$$\begin{aligned} H &= E_0 + \sum_n (\varepsilon_n - \mu) a_n^\dagger a_n + \sqrt{N_0}(\varepsilon_n - \mu)(a_n^\dagger + a_n) \\ &+ \frac{1}{2}\sum_{m,n} J_{mn}(a_m^\dagger + a_m)(a_n^\dagger + a_n) \\ &+ \lambda \sum_{l,m,n} K_{lmn}(a_l^\dagger a_m a_n + a_l^\dagger a_m^\dagger a_n) \\ &+ \lambda^2 \sum_{k,l,m,n} L_{klmn} a_k^\dagger a_l^\dagger a_m a_n \end{aligned} \tag{10}$$

where

$$E_0 = N_0 \varepsilon_0 - \frac{1}{2} N_0^2 g \int u_0^4(x)\, d^3x - \frac{1}{2}\sum_n J_{nn} \tag{11}$$

and $L_{klmn} = gN_0 \int u_k(x) u_l(x) u_m(x) u_n(x)\, d^3x$

with $J_{mn} = L_{oomn}$ and $K_{lmn} = L_{0lmn}$.

EZAWA'S PERTURBATION APPROACH

Ezawa's perturbation theory is formulated to solve for the effective Hamiltonian

$$\Lambda_n = \lambda^{-1}\Lambda_n^{(-1)} + \Lambda_n^{(0)} + \lambda\Lambda_n^{(1)} + \lambda^2\Lambda_n^{(2)} + \ldots \tag{12}$$

in

$$H\psi_n = \psi_n \Lambda_n \ , \tag{13}$$

where

$$\psi_n = A_n |n\rangle = \left(1 + \lambda A_n^{(1)} + \lambda^2 A_n^{(2)} + \ldots\right)|n\rangle \tag{14}$$

with A_n being an operator in the Hilbert space $H_B H_C$ of the total system, and Λ_n is an operator in H_C.

The perturbation problem is formulated by dividing H into three parts: H_C, which acts only on the condensate; H_B, which acts only on the out-ofcondensate particles; and H_{BC}, which involves the interaction between condensate and out-of-condensate particles. The unperturbed Hamiltonian is then taken to be the terms down to the zero-th order term in H_B and H_C, given the names H_B and H_C. The rest of the terms are taken as the perturbation. We note that H_{BC} contains a zero-order term in λ.

In the lowest order, Ezawa et al. (1998) got $\Lambda^{(-1)} = 0$, and

$$\Lambda^{(0)} = W_0 + \frac{1}{2} J_{00}(a_0^\dagger + a_0) = W_0 + J_{00} x_0^2 \ , \tag{15}$$

where $x_0 = \frac{1}{\sqrt{2}}(a_0^\dagger + a_0)$. Higher order terms are obtained after diagonalizing H_B, a process which involves only an orthogonal transformation to handle mutual interaction between out-of-condensate particles. Using the perturbation formula

$$\Lambda^{(2)} = H.P.\langle 0|VA^{(1)}|0\rangle + \left[H_C, \langle m|A^{(1)}|0\rangle\right],$$

the result obtained was

$$\Lambda^{(2)} = \frac{p_0^2}{2M_2} + \frac{p_0^4}{2M_4} + K_3 x_0^3 + K_4 x_0^4 + L_{12}\left(x_0 p_0^2 + p_0^2 x_0\right) + L_4\left(x_0 p_0 + p_0 x_0\right) \quad (16)$$

where $p_0 = -i\frac{1}{\sqrt{2}}\left(a_0 - a_0^\dagger\right)$.

NEW PERTURBATION METHOD

To investigate the higher order terms due to $H_{BC}^{(0)}$, we shall assume that the perturbation consists of only this term $V = H_{BC}^{(0)}$, so that

$$H = H_B + H_C + V. \quad (17)$$

The perturbation approach is obtained by writing

$$\psi_n = (1+L)|n\rangle + \sum_{i\neq n}|i\rangle L_i = (1+L)P|n\rangle + K(1+L)|n\rangle$$

where

$$K|n\rangle = \sum_{i\neq n}|i\rangle L_i' = \sum_{i\neq n}|i\rangle L_i (1+L)^{-1} \ . \quad (18)$$

P is the projector to the condensate factor. We now define

$$\psi_n' = \psi_n (1+L)^{-1} = |n\rangle + \sum_{i\neq n}|i\rangle L_i' = P|n\rangle + K|n\rangle \ . \quad (19)$$

The effective Hamiltonian can now be replaced by

$$\Lambda_n' = (1+L)\Lambda_n (1+L)^{-1} \quad (20)$$

which satisfies the equation

$$H\psi_n' = \psi_n'\Lambda_n' \ . \quad (21)$$

From this new eigenvalue equation, we break it up into two parts by projecting it with respect to P and $Q = 1 - P$, giving

$$P(W_n + H_C)(P+K)|n\rangle = P(P+K)\Lambda_n'|n\rangle - PV(P+K)|n\rangle \quad (22)$$

and $Q(H_B + H_C)(P+K)|n\rangle = QKP\Lambda_n'|n\rangle - QV(P+K)|n\rangle.$

$$(23)$$

This means that we can simplify the problem by finding a perturbation operator $K = QKP$ satisfying corresponding operator equation

$$P(W_n + H_C)(P+K) = P(P+K)\Lambda_n' - PV(P+K) \quad (24)$$

and

$$Q(H_B + H_C)K = QKP\Lambda_n' - QV(P+K) \quad (25)$$

The P equation can be simplified into

$$P(W_n + H_C)P = P\Lambda_n' - PV(P+K) \quad (26)$$

which allows Λ_n' to be solved for once K is found. Furthermore, QKP can be left multiplied into this equation to give

$$QKP(W_n + H_C)P = QKP\Lambda_n' - QKPV(P+K) \quad (27)$$

from which the term containing Λ_n' can be eliminated with Eq. (25), yielding

$$Q(H - W_n)K + Q[H_C, K]P = (K - Q)V(P+K) \quad (28)$$

The role of this equation is to determine K perturbatively, whether Eqs. (26) and (28) are the working equations of this perturbation approach.

RESULTS OF PERTURBATION

Using this new approach to perturbation, we get

$$V = \sum_{n\neq 0} J_{n0}\left(a_n^\dagger + a_n\right)\left(a_0^\dagger + a_0\right) \quad (29)$$

the first order result of Ezawa et al.,

$$K^{(1)} = -\frac{Q}{H_B - W_0}VP. \quad (30)$$

In fact we see that K will be a polynomial in $\left(a_0^\dagger + a_0\right) = \sqrt{2}x_0$ so that the term $Q[H_C, K]P$ vanishes to all orders, and the equation that determines K simplifies to

$$K = \frac{Q}{W_0 - H_C}(Q - K)V(P + K) \tag{31}$$

which is similar in form to the results of regular perturbation theory in operator form developed by Speisman (1957), and, therefore, we immediately get

$$K^{(2)} = \frac{Q}{W_0 - H_B}\left[VK^{(1)} - K^{(1)}V\right]$$

$$K^{(2)} = \frac{Q}{W_0 - H_B}V\frac{Q}{W_0 - H_B}VP \tag{32}$$

and

$$\begin{aligned} K^{(3)} &= \frac{Q}{W_0 - H_B}\left[VK^{(2)} - K^{(2)}VP - K^{(1)}VK^{(1)}\right] \\ &= \frac{Q}{W_0 - H_B}\left[V\frac{Q}{W_0 - H_B}V\frac{Q}{W_0 - H_B}VP\right. \\ &\quad \left. - \frac{Q}{W_0 - H_B}VPV\frac{Q}{W_0 - H_B}VP\right] \end{aligned} \tag{33}$$

Finally, the new Hamiltonian is given by substituting these expressions for K into Eq. (26). Explicitly, $\Lambda^{(2)}$ is proportional to x_0^2 and $\Lambda^{(4)}$ proportional to x_0^4. Since x_0 was estimated to be a small quantity in Ezawa's work, we see that indeed, we have explicitly verified that the procedure to include $H_{BC}^{(0)}$ in V is justified.

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