Deformation Quantization and Representations of the Real Rotation Group

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ABSTRACT

In this paper we present concrete computations of representations of the real rotation group $SO(3,\mathbb{R})$ arising from deformation quantization of its coadjoint orbit $\Omega_\xi \cong S^2$. In particular, we construct the quantization mapping ℓ,

$$\ell_S \quad i f_S \star : C^\infty(\Omega_\xi)[[\lambda]] \longrightarrow C^\infty(\Omega_\xi)[[\lambda]].$$

where ℓ is a is a Lie algebra representation of $\mathfrak{g} = so(3,\mathbb{R})$

$$\mathfrak{g} \ni S \mapsto \ell_S \in \mathrm{End}\,(C^\infty(M)[[\lambda]]).$$

We will prove that the left-regular representation T of $SO(3,\mathbb{R})$ is just $T = \exp(\hat{\ell})$, where $\hat{\ell}$ is unitarily equivalent to ℓ. So instead of the usual Hilbert space of representation, $\mathcal{H}$, we need to have an associative algebra

$$\mathcal{A} = (C^\infty(M)[[\lambda]], \star_\lambda)$$

which gives a deformation of the commutative algebra and Lie algebra structures of $C^\infty(M)$. $\mathcal{A}$ is called a deformation quantization of the symplectic manifold M. The requirement that ℓ be a Lie algebra representation is that the equation $i f_S \star i f_T - i f_T \star i f_S = \{ i f_S, i f_T \} = i f_{[S,T]}$, $(S,T \in \mathfrak{g})$ should be satisfied. This means that the coadjoint action of the real rotation group on its orbits should be strictly homogeneous, and furthermore, that the star-product $\star_\lambda$ is a covariant star-product, i.e., the Lie subalgebra

$$\mathfrak{h} := \mathrm{span}\,\{ f_S : S \in \mathfrak{g} \}$$

of $C^\infty(M)$ is an $\mathfrak{h}$-relative quantization with respect to $\star_\lambda$. An appropriate Fourier transform intertwines ℓ_S and a differential operator $\hat{\ell}_S$. It is proved that $\hat{\ell}_S$ equivalent to the action of rotation given by the element $\exp S$ of the rotation group $G = SO(3,\mathbb{R})$, that is, $\hat{\ell}_S$ is the differential of the left regular representation. Exponentiation of the representation $\hat{\ell}_S$ gives the corresponding representations of G.

Key words: deformation quantization, orbit method, coadjoint orbit, symplectic manifold, representation theory, rotation group

INTRODUCTION

Quantization is a process by which quantum systems are assigned to classical mechanical systems. To illustrate, n free particles are classically described by $2n$ coordinates $(p_1, ..., p_n, q^1, ..., q^n)$ where the p_j's are momenta and the q^i's are the position coordinates. H. Weyl (1931) gave the following prescription for the quantization of this system:

$$q^i \rightarrow \hat{q}^i = \text{multiplication by } q^i,$$

$$p_j \rightarrow \hat{p}_j = \frac{-i}{\hbar} \frac{\partial}{\partial q^j},$$

where the operators on the right hand side act on Hilbert space $L^2(\mathbb{R}^n)$. The theory of quantum mechanics requires that the correspondence principle holds:

$$[\hat{p}_j, \hat{q}] = i\hbar \, \widehat{\{p_j, q^i\}} = \delta^i_j I,$$

where { , } is the Poisson bracket on $\mathbb{R}^{2n}$.

In general, quantization means the assignment of Hilbert space operators to functions on phase space. Recall that the Hamiltonian formulation of classical mechanics has for its framework a symplectic manifold (M,ω). The classical observables are smooth real functions on M, denoted by $C^\infty(M)$, and the evolution of observables satisfies the differential equation

$$\frac{d}{dt} f_t = \{H, f_t\}$$

Here $H \in C^\infty(M)$ is the Hamiltonian (e.g., energy) of the system.

On the other hand, Heisenberg's formulation of quantum mechanics considers a Hilbert space $\mathcal{H}$, the quantum observables being self-adjoint operators on $\mathcal{H}$. The time evolution of observables satisfy

$$\frac{dA_t}{dt} = \frac{-i}{\hbar}[H, A_t] = H \circ A_t - A_t \circ H,$$

where the Hamiltonian H is a self-adjoint operator on $\mathcal{H}$.

A natural definition for quantization is that it is a linear mapping

$$Q : C^\infty(M) \rightarrow \{\text{self-adjoint operators on } \mathcal{H},\}$$

such that

(Q1) $Q(1) = Id_{\mathcal{H}}$,

(Q2) $Q(\{f, g\}) = \frac{-i}{\hbar} [Q(f), Q(g)]$.

The requirement (Q2) means that the Lie algebra structure of functions on phase space under Poisson bracket goes over to the Lie algebra structure of operators under commutators. Moreover (Q2) limits the class of functions that can be "quantized," that is, there is no correspondence Q defined on all of the smooth functions on M when irreducibility requirements are imposed. The fact that it is not possible that all elements of $C^\infty(M)$ may be made to correspond to self-adjoint operators and still satisfy Heisenberg's correspondence principle has been known for quite a while as Groenewold-Van Hove theorem (Abraham & Marsden, 1978; Groenewold, 1946; Van Hove, 1951). In the example given above, the quantizable functions are those which belong to some symbol class. Mathematical theories addressing the"irreducibility" problem are the geometric quantization theory of Kostant & Souriau (Kostant, 1970; Auslander & Kostant, 1971; Kirillov, 1962; Kirillov, 1976), Berezin's quantization (Berezin, 1975, Berezin 1974) and deformation quantization. The idea of quantization by deformation of structures of the algebra of classical observables was proposed by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in the mid 1970's (Bayen et al., 1977; 1978). It consists of replacing operators on Hilbert space as the quantum observables by formal power series in some variable λ, and interpreting the correspondence principle as an equality only up to second order in λ. Indeed, Bayen and others suggested that:

> "quantum mechanics be interpreted as a deformation of the algebra of observables, and not as a radical change in the nature of observables."

As was mentioned above, there is no quantization of $\mathbb{R}^{2n}$, i.e. , there is no linear mapping $f \rightarrow Q(f)$ satisfying

(Ql) and (Q2). One does not even have a quantization of the polynomial algebra on $\mathbb{R}^{2n}$ as symmetric operators in Hilbert space for which q^i and p_j are represented irreducibly (Groenewold-Van Hove theorem).

Quite recently an analogue of this theorem was found for the sphere S^2 (Gotay et al., 1996) The following list apparently points to some general obstruction phenomena to quantization, that is, the examples satisfy a Groenewold-Van Hove type theorem:

1. nilpotent basic algebras on connected symplectic manifolds;

2. T^*S^1 (the cylinder $\mathbb{R} \times S^1$), with basic algebra $e(2)$ which is the Lie algebra of the solvable group $E(2)$ (Euclidean group);

3. basic Lie algebras on compact symplectic manifolds, in particular, S^2;

4. finite-dimensional quantizations of basic Lie algebras on noncompact symplectic manifolds, in particular, $\mathbb{R}^{2n}$ with Lie algebra $\mathfrak{h}(2n)$ (Heisenberg Lie algebra).

In Gotay (1995); Gotay & Grundling (1997a, 1997b); Gotay & Grabowski (1999); Gotay, Grabowski, & Grundling (1997); Gotay, Grundling, & Hurst (1995) and Gotay, Grundling, & Tuynman (1998) the full details and the precise meaning of basic algebra are discussed.

The conjecture of the existence of a general obstruction to quantization is false, however, as Gotay and co-workers showed. They demonstrated that there is no obstruction to quantization of the torus $\mathbb{T}^2$. Furthermore, as item 3 in the case of the Lie algebra $e(2)$ indicates, a corresponding obstruction should appear for basic solvable Lie algebras, in analogy with item 2. This is not true since there is an infinite-dimensional quantization of $T^*\mathbb{R}_+ \cong \mathbb{R} \times \mathbb{R}_+$ with affine basic algebra aff ($\mathbb{R}$). The paper by Gotay, Grundling, & Tuynman (1998) discusses at length the circumstances under which obstructions to quantization appear and why no such obstruction occur for the examples just mentioned.

The aim of this paper is to present concrete computations of the representations of $SO(3,\mathbb{R})$, in particular the left-regular one, coming from deformation quantization of S^2, which appear as a (semisimple) coadjoint orbit. In particular, we will show that the quantization mapping $if_S\star : C^\infty(M)[[i]] \to C^\infty(M)[[i]]$ is in fact the differential of the left-regular representation of $SO(3,\mathbb{R})$ in S^2. To perform these computations, one needs to look for an appropriate coordinate system on the sphere that reflects certain properties of the Lie group action on it. The polar coordinate system on S^2 is a natural candidate, but it does not reflect these properties of the group action. The coordinate system arising from geodesics on S^2 satisfies the requisite properties, this time on the universal covering, but the ensuing integrability of the Lie algebra representation presents a problem and requires extra care in computing. Thus, the application of deformation quantization to the representation theory of concrete Lie groups is dependent on the proper choice of coordinate systems on coadjoint orbits. The quantization we obtain is actually a prequantization (quantization sans the irreducibility requirement), and this is consistent with the Groenewold-Van Hove result for S^2. The final step of finding the irreducible representations from among those constructed does not come from deformation quantization but from geometric quantization or the theory of induced representations. The computations performed here, it is hoped, should hint to further methods on how to treat the general case of compact semisimple Lie groups, in, particular $SO(n, \mathbb{R})$, and also the discrete series of the noncompact semisimple classical groups. It is difficult to say the same thing for the continuous series of the noncompact groups since they require a considerable modification of the orbit method, for example, the representation space of the principal continuous series of $SL(2, \mathbb{R})$ is not a coadjoint orbit in the usual sense. Ideally, each of the more important considerations in representation theory (namely, characters, intertwining operators, induction, restriction, and others), or some suitable modification thereof, should be defined in terms of deformation quantization. This has been done for the case of connected, simply connected nilpotent groups and exponential groups but is not yet complete for solvable groups (Arnal 1984; Arnal & Cortet 1990a; 1990b; 1985; Arnal et al., 1983) and semisimple groups. The framework with which we shall work within is the so-called method of orbits introduced by A. A. Kirillov in 1962. This is the framework employed in the articles by Arnal (1978), Arnal & Cortet, (1985, 1990a, 1990b), and Arnal et al., (1990). In the orbit method the basic object is a coadjoint orbit of a Lie group G in the dual space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$ = Lie(G) .

The application of deformation theory to the theory of Lie group representations has been pursued since the mid-1970s (Fronsdal, 1989; Bayen et al., 1977) although only a few results have been obtained. It was only in the early to mid-1980s that a better understanding of the application came to light. What is needed is an appropriate notion of group invariance of the star-product. In Bayen et al., (1977) the notion of geometrical invariance was introduced. However, it finds only limited application since almost all important examples of symplectic manifolds with Lie group action do not have this invariance property. A correct notion of invariance, called covariance, is investigated by Arnal et al., (1983). The properties of covariance is compared with those of other invariance concepts, and its applicability to representation theory of nilpotent Lie groups commenced. Arnal (1984) gave the first concrete application of deformation quantization to representation theory. The Lie groups considered consisted of the class of connected, simply connected nilpotent Lie groups and works within the framework of the orbit method of Kirillov (1962). Further investigations into this same class of Lie groups followed (Arnal & Cortet, 1990a), where other familiar considerations in representation theory, such as Fourier transform and group C^*-algebras (Gelfand-Naimark-Segal construction), are defined in terms of or found connections with deformation quantization. Arnal & Cortet (1985) and Arnal et al. (1995) continued this program to exponential and type I solvable Lie groups, respectively.

In short articles by Do Ngoc Diep & Nguyen Viet Hai (1999a; 1999b), Nguyen Viet Hai (2000a; 2000b), and Arnal & Cortet (1990b), the techniques that appear in the papers mentioned previously are applied to concrete Lie groups, namely the motion group $E(2)$, the real and complex affine groups Aff($\mathbb{R}$) and Aff($\mathbb{C}$), and the diamond groups introduced by Do Ngop Diep (1999). We note, for example, that Aff($\mathbb{R}$) is solvable but not type I. Indeed these papers attempt to apply deformation quantization to other classes of Lie groups. Moreover they allow one to see clearly what is going on. We remark further that the results in these papers are consistent with the works of Gotay and coworkers. For example, there is a complete description of the list of infinite-dimensional unitary irreducible representations of Aff($\mathbb{R}$) (Do Ngoc Diep & Nguyen Viet Hai, 1999a) in terms of deformation quantization of the coadjoint orbit $\mathbb{R}\times\mathbb{R}_+$, although the authors seem to have forgotten about the series of one-dimensional irreducible representations. The paper by Arnal & Cortet (1990b) gives a description of unitary, not necessarily irreducible, representations of $E(2)$ in terms of the deformation quantization of the cylinder $T^* S^1$.

DEFORMATION QUANTIZATION

Definition 1 *Let* (M,ω) *be a symplectic manifold. By deformation quantization of* (M,ω) *we mean an associative algebra structure on the space* $C^\infty(M)\,[[\lambda]]$ *of formal power series, in the variable* λ *with coefficients in* $C^\infty(M)$, *with respect to some product* $\star_\lambda . \star_\lambda$ *is called a star-product and we shall write* $\star_\lambda$*-product. The* $\star_\lambda$*-product has the form*

$$a \star_\lambda b = \left(\sum a_s \lambda^s\right) \star_\lambda \left(\sum b_t \lambda^t\right) = \sum c_k \lambda^k \qquad (1)$$

where

(DQ_1) *the coefficients* $c_k = c_k(a,b)$ *depends not only on a and b but also on the partial derivatives* $\partial_i^\alpha a, \partial_j^\beta b$ *where* $i + j + |\alpha| + |\beta| \le k$,

(DQ_2) $c_0 = a_0 b_0$,

(DQ_3) $c_1(a,b) - c_1(b,a) = \lambda\{a,b\}$, for $a, b \in C^\infty(M)$.

(DQ_1) means that the $\star_\lambda$-product is local. This is also equivalent to the condition that the c_k are given by differential operators. (DQ_2) means that the $\star_\lambda$-product is a deformation of the commutative pointwise product of functions in $C^\infty(M)$. Lastly, by defining the Lie bracket

$$[a, b]_{\star_\lambda} = \frac{1}{2\lambda}\,(a \star_\lambda b - b \star_\lambda a),$$

(DQ_3) means that the $\star_\lambda$ generates a deformation of the Poisson bracket { , } on $C^\infty(M)$.

A more general definition of deformation quantization may be given for Poisson manifolds M, manifolds with a contravariant skew-symmetric 2-tensor. The main difference with symplectic manifolds is that the bilinear mapping induced on $C^\infty(M)$ may be degenerate. We shall, however, consider only symplectic manifolds in this paper .

Example The basic example here is that of the symplectic manifold ($\mathbb{R}^{2n}$, $\omega = \sum dp^i \wedge dq_i$), which has for its deformation quantization the algebra ($C^\infty(\mathbb{R}^{2n})$ $[[ih/2]]$, $\star_M$) where,

$$u \star_M v = \exp\left(\frac{ih}{2}\,\omega^{-1}\ \partial_r u\, \partial_r v\right)$$

$$= \sum \frac{ih}{2}^r \frac{1}{r!}\,\omega^{i_1 j_1} \ldots \omega^{i_r j_r}\ \frac{\partial_r u}{\partial x^{i_1} \ldots \partial x^{i_r}}\ \frac{\partial_r v}{\partial x^{j_1} \ldots \partial x^{j_r}}$$

$$= u \cdot v + \text{terms of higher order in } ih,$$

where $u, v \in C^\infty(\mathbb{R}^{2n})$, $(x^1,\ldots,x^{2n}) = (p^1,\ldots,p^n, q_1,\ldots,q_n)$, and $(\omega^{ij}) = \begin{matrix} 0 & I_n \\ -I_n & 0 \end{matrix}$. The Lie bracket between u and v *is*

$$[u, v\,]_{\star_M} = \frac{1}{2i} \sinh\left(\frac{ih}{2}\,\omega^{-1}\, \partial_r u\, \partial_r v\right)$$

$$= \{u,v\} + \text{terms of higher order in } (ih)^2,$$

showing the deformation of the Poisson bracket.

The symbol $\star_M$ is read as Moyal star-product (Moyal, 1949). It is intimately connected with the Weyl quantization of $\mathbb{R}^{2n}$. For functions a on $\mathbb{R}^{2n}$ belonging to symbol class (Fedosov, 1993) there corresponds the operator $\hat{a} = A$ acting on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$

$$(Au)(q) = \int_{\mathbb{R}^{2n}} exp\left(\frac{i}{h} p(q\text{-}q')\right)\ a\left(\frac{q+q'}{2}, p\right) u(q')\,dq'dp$$

This is actually the formula which gives the correspondence given in the introduction, i.e., $q_i \mapsto \hat{q}_i$ $p^j \mapsto \hat{p}^j$. The Moyal $\star$-product enters into the formula for the composition of operators

$$\hat{a} \circ \hat{b} = \widehat{a \star_M b}$$

where a and b are symbols.

The Moyal $\star$-product will be very important for us in applications to the representation theory of Lie groups.

We introduce next the notion of covariance of a star-product.

Definition 2 *Let* (M,ω) *be a symplectic manifold. A subalgebra* $\mathfrak{h}$ *of* $C^\infty(M)$ *is called an* $\mathfrak{h}$*-relative quantization with respect to the star-product* $\star_\lambda$ *if*

$$[u, v]_{\star_\lambda} \quad \frac{1}{2\lambda}(u \star_\lambda v - v \star_\lambda u) = \{u, v\}, \qquad u, v \in \mathfrak{h}.$$

Definition 3 *Let the Lie group* G *act on the symplectic manifold* M *in a strictly homogeneous manner. The star-product* $\star_\lambda$ *on* $C^\infty(M)$ *is said to be a covariant star-product if*

$$[\lambda f_S, \lambda f_T]_{\star_\lambda} = \lambda\{ f_S , f_T\}$$

for all generating functions $f_S, f_T, S, T, \in \mathfrak{g} = \text{Lie}(G)$.

The point of these definitions is that the operator of left $\star_\lambda$-multiplication

$$\ell_S \quad \lambda f_S \star : C^\infty(M)[[\lambda]] \to C^\infty(M)[[\lambda]],$$

$$\ell_S(u) = \lambda f_S \star_\lambda u,\ u \in C^\infty(M)[[\lambda]]$$

is a Lie algebra representation of $\mathfrak{g}$. It remains to find some equivalent expression (e.g., differential operators), if possible, for ease of computations and, more importantly, to find invariant subspaces. Exponentiation gives unitary representations of the corresponding Lie group of $\mathfrak{g}$.

Deformation quantization (of Poisson, symplectic, Kahler manifolds, etc.) itself, i.e., without considering the representation theory of symmetry groups of the manifolds concerned, is of course of separate interest.

Renewed interest in deformation quantization came along because of the works of Drinfeld on quantum groups (which are deformation of Lie groups), and because of B. Fedosov's solution to the problem of deformation quantization of symplectic manifolds, which employed only geometric concepts, and did not make any cohomological constructions. The corresponding and considerably more difficult deformation quantization problem for Poisson manifolds was solved fairly recently by M. Kontsevich (1997). Fedosov's book (1993) deals with the formulation of an index theory using deformation quantization. The works of Karabegov (1996) and Reshetikhin & Takhtajan (1999) and others, employed similar methods as those discovered by Fedosov to solve the corresponding deformation quantization of Kahler manifolds and other special classes of complex manifolds. On the other hand M. Rieffel (1994) gives stricter criteria for deformation quantization which takes into account C^*-algebra structures on manifolds, in particular Heisenberg manifolds.

HAMILTONIAN MECHANICS AND COADJOINT ORBITS OF *SO* (3,$\mathbb{R}$)

Throughout the rest of this paper we let $G = SO\,(3, \mathbb{R})$, $\mathfrak{g} = so(3,\mathbb{R})$ and $\mathfrak{g}^* = so(3,\mathbb{R})^*$.

For semisimple Lie subgroups of the full linear group $GL(n,\mathbb{R})$, the coadjoint representation takes the following form:

$$Ad\,(g)S = gSg^{-1},\ g \in G, S \in \mathfrak{g}$$

There is a nondegenerate bilinear form on $\mathfrak{g}$, namely, $< S,T >=$ trace $(S \bullet T)$. Because of this and the fact that G is semisimple, we have that $\mathfrak{g}^* \cong \mathfrak{g}$. (This is not true, say, for nilpotent Lie groups). Therefore the coadjoint representation $K : G \to Aut\,(\mathfrak{g}^*)$ is given by

$$\mathrm{K}(g)\xi = g\xi g^{-1},\quad \xi \in \mathfrak{g}^*,$$

and the determination of coadjoint orbits is equivalent to describing conjugacy classes.

Choose and fix $\xi \in \mathfrak{g}^*$. We may look at G_ξ as the group of rotations of 3-space fixing the axis determined by the vector $\xi \in \mathfrak{g}^* \cong \mathfrak{g} \cong \mathbb{R}^3$. This means $G_\xi \cong SO(2,\mathbb{R})$ and hence $G/G_\xi \cong SO(3,\mathbb{R}) \cong SO(2,\mathbb{R}) \cong \Omega_\xi$. It is well-known that $SO(n+1,\mathbb{R})/\,SO(n,\mathbb{R}) \cong S^n$, the unit n-sphere. Thus $\Omega_\xi \cong S^2$.

Now, coadjoint orbits are symplectic manifolds (Kirillov, 1962), in which case Hamiltonian mechanics may be performed on them. If there is a Lie group action on a symplectic manifold M by symplectomorphisms, that is, for each $g \in G$

$$g^*\omega = \omega,\quad g \in G \text{ (pullback of the action)}$$

we call M a homogeneous symplectic manifold and there is a structure of a G-module on $C^\infty(M)$ defined by

$$(g \bullet u)(\xi) = u\ (g^{-1} \bullet \xi)$$

where $g \in G$, $\xi \in M$, $u \in C^\infty(M)$. Derivation gives a $\mathfrak{g}$-module structure on $C^\infty(M)$ and this is given by the following: To each $S \in \mathfrak{g} = Lie\,(G)$ of the Lie algebra of G there corresponds a Hamiltonian vector field $\eta_S \in$ Vect (M) given by

$$(\eta_S u)(\xi) = \frac{d}{dt}\, u\ (\exp(-tS) \cdot \xi)\Big|_{t=0}$$

The module structure is now given by $(S \cdot u)(\xi) = (\eta_S u)(\xi)$. If the η_S are strictly Hamiltonian and the generating functions f_S and f_T satisfy

$$f_{[S,T]} = \{ f_S , f_T \}$$

we call M a strictly homogeneous symplectic manifold. The generating function f_S is defined as the solution of the differential equation

$$i(\eta_S)\omega = -d f_S .$$

The following short exact sequence of Lie algebras reflects the strictly Hamiltonian character of the action of G on M.

$$0 \to \mathbb{R} \to C^\infty(M,\mathbb{R}) \to \mathfrak{g} \to 0 \qquad (2)$$

Although the mapping $S \mapsto \eta_S$ is a Lie algebra representation of $\mathfrak{g}$, it does not give us a quantization since η_S sends all constants to zero. Instead we work on the space of generating functions f_S and perform a deformation quantization of this space. A very important class of strictly homogeneous symplectic manifolds are coadjoint orbits of a connected Lie group G in the dual space $\mathfrak{g}^*$. In fact, we know the following from Kostant:

Theorem 1 *(Kostant) Any symplectic manifold M with a Hamiltonian G-action, where G is a connected Lie group, is locally isomorphic to a coadjoint orbit of G or a real central extension $\tilde{G}$ of G.*

This theorem implies that the coadjoint orbits are all the classical systems that we need.

QUANTIZATION OF THE SPHERE

A criteria in choosing the appropriate coordinates in (Ω_ξ, ω_ξ) is that it must reflect the strictly G-homogeneous action of the group G on Ω_ξ, i.e.,

$$\{ f_S,\ f_T\} = f_{[S,\,T]}\,,$$

for generating functions $f_S, f_T \in C^\infty(M)$. Another criteria is the relation of relative quantization or the covariance of star-product

$$[\lambda f_S, \lambda f_T\,]_{\star\lambda} \quad \frac{1}{2\lambda}(\lambda f_S \star \lambda f_T - \lambda f_T \star \lambda f_S) = \lambda\{f_S,\ f_T\}.$$

With respect to the polar coordinate system

$$\tau\colon \mathbb{R}^2 \quad (p,q) \mapsto (\sin p \sin q, \sin p \cos q, \cos p) \in S^2 \cong \Omega_\xi,$$

the Moyal star-product is not a covariant star-product. This means that the Lie algebra of generating functions $\tilde{\mathfrak{g}}$ does not have a relative quantization with respect to this star-product. However, this coordinate system is interesting since it provides a deformation quantization of the algebra $C^\infty(S^2)$ arising from the Moyal $\star$-product on $\mathbb{R}^2$. Since the polar coordinate system is global, $\star$-products of functions hold globally (up to linear transformations). A parametrization which suits the requirement of covariance will now be explained. This time, however, the action of G on Ω_ξ is not strictly G-homogeneous, in which case, we use theorem 1 which states that it is the real central extension $\tilde{G}$ of G which acts on the simply connected covering $\tilde{\Omega}$ of Ω_ξ. in a strict G-homogeneous manner.

Fix the functional $\xi = X^* \in \mathfrak{g}^* = so(3,\mathbb{R})^*$. The coordinate system that we need is the one coming from the exponential mapping $\exp : T_{X^*}(\Omega_{X^*}) \to \Omega_{X^*}$, where $T_{X^*}(\Omega_{X^*})$ is the tangent space to Ω_{X^*} at X^*. Since $T_{X^*}(\Omega_{X^*}) \cong T_{X^*}(S^2) \cong \mathbb{R}^2$, exp will give a local diffeomorphism $\varphi\colon\mathbb{R}^2 \to \Omega_{X^*}$. This local diffeomorphism is given

$$\varphi(p,q) := \exp(pY^* + qZ^*)$$

$$= (\cos 1)X^* + (\sin 1)(pY^* + qZ^*)$$

We modify this mapping and use the simpler

$$\varphi(p,q) := Z^* + pY^* + qZ^*$$

replacing cos 1 and sin 1 with 1. The effect is that the symplectic form ω appearing below will not have the factor $\sin^2 1$.

The symplectic form on Ω_{X^*} is now given by

$$\omega(\eta_S \otimes \eta_T) = \langle X^*, [S,T] \rangle.$$

The Hamiltonian function f_S associated to $S = \alpha_1 X + \beta_1 Y + \gamma_1 Z \in \mathfrak{g}$ is given by

$$f_S(p,q) = \langle \xi, S \rangle = \langle X^* + pY^* + qZ^*, \alpha_1 X + \beta_1 Y + \gamma_1 Z \rangle$$

$$= \alpha_1 + \beta_1 p + \gamma_1 q$$

The Hamiltonian vector field η_S is given by

$$\eta_S = \beta_1 \frac{\partial}{\partial q} - \gamma_1 \frac{\partial}{\partial p}\,.$$

Indeed,

$$\eta_S(f) = \frac{\partial f_S}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial f_S}{\partial q}\frac{\partial f}{\partial p}\,,$$

the right-hand side being the Poisson bracket in $C^\infty(\mathbb{R}^2)$.

Write $\omega = \varphi^* \omega_{X^*}$ and let $T = \alpha_2 X + \beta_2 Y + \gamma_2 Z$ be another element of $\mathfrak{y}$. Then $f_T = \alpha_2 + \beta_2\, p + \gamma_2\, q$ and $\eta_T = \beta_2 \frac{\partial}{\partial q} - \gamma_2 \frac{\partial}{\partial p}$. We have

$$<\omega,\ \eta_S \otimes \eta_T> = <\omega,\ (\beta_1\gamma_2 - \beta_2\gamma_1)\frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q}\ + \ldots>$$

$$= \beta_1\gamma_2 - \beta_2\gamma_1$$

$$= <X^*,\ (\beta_1\gamma_2 - \beta_2\gamma_1)\, X + \ldots>$$

$$= <X^*, [S, T]> = <\omega_{X^*},\ \eta_S \otimes \eta_T>.$$

Therefore, we conclude that $\omega = \varphi^* \omega_{X^*} = dp \quad dq$.

Before proceeding, let us first summarize our discussion.

Theorem 2

1. *The exponential mapping* exp: $T_{X^*}(\Omega_{X^*}) \to \Omega_{X^*}$ gives rise to a local diffeomorphism (in fact, a symplectomorphism) $\varphi : \mathbb{R}^2 \to \Omega_{X^*}$ given by

$$\varphi\,(p,q) = X^* + pY^* + qZ^*.$$

2. *Under the mapping* φ *in* 1., *the Hamiltonian function* f_S *associated to a vector* $S = \alpha X + \beta Y + \gamma Z \in \mathfrak{y}$ has the form

$$f_S\,(p,q) = \alpha + \beta p + \gamma q;$$

the Hamiltonian vector field $\eta_S \in$ Vect$\{\ \Omega_{X^*})$ has the form

$$\eta_S = \beta\,\frac{\partial}{\partial q} - \gamma\frac{\partial}{\partial p}\ ;$$

and the Kirillov symplectic form ω *is*

$$\omega = \varphi^* \omega_{X^*} = dp \qquad dq.$$

THE OPERATOR $\hat{\ell}_S$

The action of the real rotation group G on Ω_{X^*} is only Hamiltonian (or homogeneous) and not strictly homogeneous on account of the relation

$$\{ f_S, f_T\} = f_{[S,\,T]} - c\,(S,T)$$

where $c(\ .\ ,\ .)$ is some 2-cocycle. The form c disappears when we consider, instead of $\mathfrak{y}$, the Lie algebra

$$\tilde{\mathfrak{y}} = \{\tilde{S} = (S, t) : S \in \mathfrak{y}\ ,\ t \in \mathbb{R}\} \cong \mathfrak{y} \otimes \mathbb{R}$$

with Lie bracket $[(S, t_1),(T,t_2)] = ([S,T],\ c\ (S,T))$. The Lie algebra $\tilde{\mathfrak{y}}$ may be obtained in another way. It is simply the Lie algebra under Poisson bracket which is generated by generating functions f_S, $S \in \mathfrak{y}$. Indeed from the exact sequence (2) of Lie algebras

$$C^\infty(M) \cong \mathfrak{y} \otimes \mathbb{R} \cong \tilde{\mathfrak{y}}\ .$$

What this direct product means is the following. The solution of the differential equation

$$df = -i(\eta_S)\omega$$

is unique only up to a constant addend, that is, f_S and f_S + const solve the differential equation. Thus, for a vector $S \in \mathfrak{y}$, f_S and $f_{\tilde{S}}$ differ only by a constant. Although $f_S \star f_T$ and $f_{\tilde{S}} \star f_{\tilde{T}}$ differ by a constant,

$$[\ f_S\,, f_T]_\star = [f_{\tilde{S}}\,, f_{\tilde{T}}\,]_\star\ .$$

Therefore the covariance property of the Moyal star-product still reads the same, where, upon choosing $\lambda = i$

$$[\ if_{\tilde{S}}\,,\ if_{\tilde{T}}]_\star = i\{\ f_{\tilde{S}}\,, f_{\tilde{T}}\}_\star = if_{[\tilde{S}\ \tilde{T}\,]}\,.$$

This means that the operator

$$\ell_S : C^\infty(M)\ [[i]] \longrightarrow\ C^\infty(M)\ [[i]],\ M = \mathbb{R}^2 \cong \Omega_{X^*}\,,$$

$$\ell_S := \ if_S \star$$

is a representation of the Lie algebra $\mathfrak{y}$. To be more precise, the operator should be $if_{\tilde{S}} \star$ and $S \mapsto \ell_S = if_{\tilde{S}} \star$ is a representation of the Lie algebra $\tilde{\mathfrak{y}} = su\,(2)$ (coming from the Lie group $\tilde{G} = SU(2)$ which is the universal covering group of $G = SO(3,\mathbb{R})$). However, the appearance of a constant ad lend merely goes into the constant multiplier appearing ın the differential operator expression for ℓ_S (to be obtained as follows).

Theorem 3 *Let $S=\alpha X+\beta Y+\gamma Z \in \mathfrak{g}$ and let $f \in C_0^\infty(\mathbb{R}^2)$ be a smooth function with compact support. Then we have*

$$\hat{\ell}_S(f) \quad \mathcal{F}_1 \circ \ell_S \circ \mathcal{F}_1^{-1}(f)$$

$$= \beta\left(\frac{1}{2}\frac{\partial}{\partial q} - \frac{\partial}{\partial x}\right)(f) + i\gamma\left(q - \frac{x}{2}\right)\cdot f + i\alpha \cdot f.$$

Here,

$$\mathcal{F}_1(f)(x,q) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ipx} f(p,q)\, dp$$

is the Fourier transform with respect to the first variable. The inverse Fourier transform is

$$\mathcal{F}_1^{-1}(f)(p,q) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ipx} f(x,q)\, dx$$

Proof of Theorem 3

$$\hat{\ell}_S \; f \quad = \mathcal{F}_1 \circ \ell_S \circ \mathcal{F}_1^{-1} \; f \quad = \mathcal{F}_1 \; i\, f_S \star \circ \ell_S \circ \mathcal{F}_1^{-1} \; f$$

$$= i\mathcal{F}_1 \; f_S \cdot \mathcal{F}_1^{-1} \; f \; + \sum_{r \geq 1} \frac{1}{r!} \left(\frac{1}{2i}\right)^r P^r \; f_S, \mathcal{F}_1^{-1} \; f$$

$$= i\mathcal{F}_1 \; f_S \cdot \mathcal{F}_1^{-1} \; f \; + \frac{1}{2i} P^1(f_S, \mathcal{F}_1^{-1} \; f$$

$$= i\mathcal{F}_1 \quad \alpha + \beta p + \gamma q \; \cdot \mathcal{F}_1^{-1} \; f$$

$$+ \frac{1}{2i}\beta \mathcal{F}_1^{-1} \frac{\partial f}{\partial q} - \gamma \frac{\partial}{\partial p} \; \mathcal{F}_1^{-1} \; f$$

$$= i\mathcal{F}_1 \; \alpha \cdot \mathcal{F}_1^{-1} \; f \; + i\beta \mathcal{F}_1^{-1} \; \frac{\partial f}{\partial x} \; + \gamma\, \mathcal{F}_1^{-1} \; q \cdot f$$

$$+ \frac{\beta}{2i} \mathcal{F}_1^{-1} \frac{\partial f}{\partial q} \; - \frac{i\gamma}{2i} \mathcal{F}_1^{-1} \; x \cdot f$$

$$= i\alpha \cdot f - \beta \frac{\partial f}{\partial x} + i\gamma q \cdot f + \frac{\beta}{2}\frac{\partial f}{\partial q} - i\frac{\gamma}{2} x \cdot f$$

$$= \beta\left(\frac{1}{2}\frac{\partial}{\partial q} - \frac{\partial}{\partial x}\right) \cdot f + i\gamma\left(q - \frac{x}{2}\right) \cdot f + i\alpha \cdot f.$$

Therefore, for $S= \alpha X + \beta Y + \gamma Z \in \mathfrak{g}$,

$$\hat{\ell}_S = \beta\left(\frac{1}{2}\frac{\partial}{\partial q} - \frac{\partial}{\partial x}\right) + i\gamma\left(q - \frac{x}{2}\right) + i\alpha,$$

which takes the form

$$\hat{\ell}_S = \beta \frac{\partial}{\partial s} + i(\gamma s + \alpha) \qquad (3)$$

upon changing to new variables $s = q - x/2$, $t = q + x/2$. The (generating) function $f_S = \alpha + \beta p + \gamma q$ is called the symbol of the differential operator $\hat{\ell}_S$.

If, instead of $\xi = X^*$, we chose $\xi = Y^*$ (resp. $\xi = Z^*$), then elements of Ω_{Y^*} (resp. Ω_{Z^*}) have the form

$$\xi(p,q) = pX^* + Y^* + qZ^*,$$

(resp. $\xi = pX^* + qY^* + Z^*$). Of course, Ω_{X^*}, Ω_{Y^*}, Ω_{Z^*} are one and the same orbit since the definition of the coadjoint representation ($K(g)\xi = g\xi g^{-1}$) and of an orbit Ω_ξ imply that the elements X^*, Y^*, Z^* of $\mathfrak{g}^*$ are conjugate to each other by rotations of the sphere. We note also that if, say, $\xi = pX^* + Y^* + qZ^* \in \Omega_{Y^*}$ and $S = \alpha X + \beta Y + \gamma Z \in \mathfrak{g}$ then

$$f_S(p,q) = \alpha p + \beta + \gamma q,$$

$$\hat{\ell}_S = \alpha \frac{\partial}{\partial s} + i(\gamma s + \beta).$$

In this connection we need to make sure that for a given element $S \in \mathfrak{g}$, $\hat{\ell}_S$ is a unique operator, independent of change of charts. For a linear symplectomorphism f, say, from Ω_{Y^*} to Ω_{X^*},

$$f \quad \varphi_X \circ \varphi_Y \; : \; \Omega_{Y^*} \to \mathbb{R}^2 \to \Omega_{X^*},$$

there corresponds a unitary operator U (Fedosov, 1993), such that

$$\hat{\ell}_S = U \circ \hat{\ell}_{(f,s)} \circ U^{-1}$$

where the operator on the left hand side is with respect to the chart Ω_{Y^*}. and the operator in the middle of the right hand side is $\hat{\ell}_S$, with respect to the chart Ω_{X^*}.

Furthermore, the <u>local</u> operator $\hat{\ell}_S$ has a unique global extension to all of $\Omega_\xi \cong S^2$ since S^2 is simply connected (monodromy theorem (Kirrilov, 1976)). The conclusion is that there is a unique operator, $\hat{\ell}_S$ up to a unitary operator U.

Finally, we make the remark that the operator $\hat{\ell}_S$, aside from allowing for the fairly simple computations below, exhibits the invariant subspace $C^\infty(\Omega_{X*}) \oplus iC^\infty(\Omega_{X*}) \subset C^\infty(\Omega_{X*})[[i]]$. In effect the Fourier transform takes care of convergence issues regarding the star-product.

We next show that the mapping $G \quad \exp(tS) \mapsto \exp(t\hat{\ell}_S) \in \mathrm{Aut}\,(L^2(S^2))$ is just the left (or right) regular representation T of the rotation group G acting on the Hilbert space $L^2(S^2)$. Restricting to the $(2\ell+1)$-dimensional (ℓ=0,1, ...) subspace of harmonic functions on the sphere gives the complete list of unitary irreducible representations of G. To do this, we first set $S = Y$ to obtain the operator $\exp(t\hat{\ell}_Y)$. Then we shift charts and, in turns, let $S = X$ and $S = Z$ to get the operators $\exp(s\hat{\ell}_X)$ and $\exp(r\hat{\ell}_Z)$. Since the group elements $\exp X$, $\exp Y$, $\exp Z$ generate G, the operators give the unitary irreducible representations upon restricting to the space of harmonic functions inside $L^2(S^2)$.

Let now $S = Y$ so that,

$$\exp(tY) = \sum_{n=0}^{\infty} \frac{(tY)^n}{n!} = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}$$

Similarly, we have $\exp(sX) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix}$

$$\text{and } \exp(rZ) = \begin{pmatrix} \cos r & \sin r & 0 \\ -\sin r & \cos r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$T(\exp(tY))\, f(x_1, x_2, x_3) =$$

$$f(x_1 \cos t + x_3 \sin t,\, x_2,\, -x_1 \sin t + x_3 \cos t).$$

This is just the action of rotation (by an angle t) in 3-space preserving the y-axis. Consider a function $f \in C^\infty(\Omega_{X*})$. We look at the restriction of $f = f(x_1, x_2, x_3)$, to the great circle

$$x_1^2 + x_3^2 = 1$$

in S^2, in which case we may write it as

$$f = f(e^{is}) \quad (e^{is} = x_1 + ix_3).$$

Consequently,

$$T(\exp(tY))\, f(e^{is}) = f(e^{i(t+s)}).$$

Now $\hat{\ell}_S = \hat{\ell}_Y = 1 \cdot \frac{\partial}{\partial s} + i \cdot 0 \cdot s = \frac{\partial}{\partial s}$, so that if we put $W = e^{i(t+s)}$, we have

$$\begin{aligned} \frac{\partial}{\partial t} T(\exp(tY))\, f(w) &= \frac{\partial}{\partial t} f(\exp tY \cdot e^{is}) \\ &= \frac{\partial}{\partial t} f(e^{i(t+s)}) \\ &= \frac{\partial W}{\partial t} \frac{\partial}{\partial W} f(W) \\ &= ie^{is}e^{it} \frac{\partial f}{\partial W} \\ &= \frac{\partial}{\partial s} f(e^{i(t+s)}) \\ &= \hat{\ell}_Y\, T(\exp(tY))\, f(e^{is}). \end{aligned}$$

Because $T(\exp(tY))\, f(w)\big|_{t=0} = f(w)$, the unique solution to the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} U(t,w) = \hat{\ell}_Y\, U(t,w) \\ U(0,w) = id \end{cases}$$

is

$$U(t,w) = \exp(t\hat{\ell}_Y)\, f(w)$$

thus

$$\begin{aligned} \exp(t\hat{\ell}_Y)\, f(w) &= T(\exp(tY))\, f(w) \\ &= f(\exp(tY) \cdot w). \end{aligned}$$

Similarly

$$\exp(s\hat{\ell}_X)\, f(w) = f\,(\exp\,(sX)\cdot w),$$

$$\exp(r\hat{\ell}_Z)\, f(w)=f\,(\exp\,(rZ)\cdot w).$$

Theorem 4 *The operators* $\exp(s\hat{\ell}_X)$, $\exp(t\hat{\ell}_Y)$, $\exp(r\hat{\ell}_Z)$, *where X, Y, Z are basis elements for the Lie algebra* $so(3,\mathbb{R})$ *provide the irreducible unitary representations of* $S0(3,\mathbb{R})$ *by restricting them to the space of harmonic functions on the sphere. These representations are exactly the representations* $T = T\Omega_{X^*}$ *associated to the orbit* $\Omega_{X^*} \cong S^2$, *in accordance with the method of orbits. More precisely, to each positive integer* ℓ, $T\Omega_{X^*}$ *is the* $(2\ell+1)$*-dimensional irreducible unitary representation on the space* $L^2(S^2)$ *of spherical functions of order* ℓ, *that is, the square integrable functions on the sphere satisfying*

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial^2\varphi}+\ell\left(\ell+1\right)f=0 \qquad (4)$$

We call this the space of harmonic functions on the sphere.

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