Perturbations of a Class of Ordinary Differential Expressions Preserving the Essential Spectrum and the Nullities

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ABSTRACT

Perturbation is used to enlargen a class of differential expressions for which the essential spectrum and the nullities can be classified. One such perturbation in the $\mathcal{L}_2$– Hilbert space for the differential expressions $\mathbf{M}_0 f$ takes the form

$$\mathbf{M}f = \sum_{l=0}^{n-1} r_l(t) f^{(l)}$$

where there exists a B such that the coefficients r_l satisfy:

$$\sup_{[x,x+1]\subset I} \int_x^{x+1} \left|\frac{r_l(t)}{s_l(t)}\right|^2 dt < B$$

INTRODUCTION

This section gives the basic definitions and notations used in this paper.

Denote by $A_n(I)$, for each positive integer n, the set of complex-valued functions f on I for which $f^{(n-1)} = D^{(n-1)} f$ exists and is absolutely continuous on every compact interval of I. Let $A_0(I) = C(I)$.

Definition 1.1

Let $\mathbf{M}$ be a differential expression of the form

$$\mathbf{M} = \sum_{k=0}^{n} a_k D^{(k)}$$

where each a_k is a complex-valued function on I.

The *maximal operator* $\mathbf{T}_1(\mathbf{M})$ *generated by* M in $\mathcal{L}_2(I)$ is defined as

$$\mathcal{D}_1 = \{f \mid f \in A_n(I) \cap \mathcal{L}_2(I), \mathbf{M}f \in \mathcal{L}_2(I)\}$$

$$\mathbf{T}_1(\mathbf{M})f = \mathbf{M}f = \sum_{k=0}^{n} a_k D^{(k)} f$$

The operator $\mathbf{T}_R(\mathbf{M})$ in $\mathcal{L}_2(I)$ is defined to be the *restriction* of $\mathbf{T}_1(\mathbf{M})$ to those $f \in \mathcal{D}_1$ which have compact support in the interior of I.

Definition 1.2

The *minimal operator generated by* $\mathbf{M}$ in $\mathcal{L}_2(I)$ denoted by $\mathbf{T}_0(\mathbf{M})$, is defined to be the minimal closed extension of $\mathbf{T}_R(\mathbf{M})$ in $\mathcal{L}_2(I)$.

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Definition 1.3

The *essential spectrum* of **M** relative to $\mathcal{L}_2(I)$ denoted by σ_e(**M**) is defined as

$\sigma_e(\mathbf{M}) = \{\lambda \in \mathbb{C} \mid \mathcal{R}(T_0(\mathbf{M} - \lambda)) \text{ is not closed}\}$

Definition 1.4

The *essential resolvent* of **M**, denoted by ρ_e(**M**), is the set of scalars not in σ_e (**M**), that is

$\rho_e(\mathbf{M}) = \{\lambda \mid \lambda \in \mathbb{C} \setminus \sigma_e(\mathbf{M})\}$

Definition 1.5

Since $\mathcal{R}(T_0(\mathbf{M}))$ is closed if and only if $\mathcal{R}(T_1(\mathbf{M}))$ is closed, then

$\sigma_e(\mathbf{M}) = \{\lambda \in \mathbb{C} \mid \mathcal{R}(T_1(\mathbf{M} - \lambda)) \text{ is not closed}\}$

Definition 1.6

Let **M** be a differential expression defined on *I*. The nullity of **M**, denoted by nul (**M**), is defined as

$\text{nul}(\mathbf{M}) := \dim(\mathcal{N}(T_1(\mathbf{M}))$

SPECIAL EXPRESSIONS

Consider the differential expression of the form

$$\mathbf{M}_0 f := \sum_{\sigma=0}^{r} a_\sigma t^{\alpha_\sigma} f^{(\rho_\sigma)} \tag{1}$$

with $r \in \mathbb{N}$, $\rho_0, \cdots, \rho_r \in \mathbb{N}_0$, and $\alpha_\sigma \in \mathbb{R}$ ($\sigma = 0, \cdots, r$) such that

$$\alpha_0 = 0, \alpha_1 \leq \rho_1 \tag{2}$$

and

$$1 \geq \frac{\alpha_\sigma - \alpha_{\sigma-1}}{\rho_\sigma - \rho_{\sigma-1}} \geq \frac{\alpha_{\sigma+1} - \alpha_\sigma}{\rho_{\sigma+1} - \rho_\sigma} \tag{3}$$

for $1, \cdots, r - 1$ if $r > 1$

Denote by $\sigma_1 < \sigma_2 < \cdots < \sigma_{s-1}$ those indices σ ($\sigma = 1, \cdots, r - 1$) for which the strict inequality holds in (3)

Then together with $\sigma_0 := 0$ and $\sigma_s := r$, we have

$$\frac{\alpha_{\sigma j} - \alpha_{\sigma j-1}}{\rho_{\sigma j} - \rho_{\sigma j-1}} = \frac{\alpha_\sigma - \alpha_{\sigma-1}}{\rho_\sigma - \rho_{\sigma-1}} \tag{4}$$

for $\sigma_{j-1} < \sigma < \sigma_j$ ($j = 1, \cdots, s$) if $s \geq 1$ and

$$\frac{\alpha_{\sigma j} - \alpha_{\sigma j-1}}{\rho_{\sigma j} - \rho_{\sigma j-1}} > \frac{\alpha_{\sigma j+1} - \alpha_{\sigma j}}{\rho_{\sigma j+1} - \rho_{\sigma j}} \tag{5}$$

for $j = 1, \cdots, s - 1$, if $s > 2$

We assume the constants $a_\sigma \in \mathbb{C} \setminus \{0\}$ to satisfy the following conditions:

$a_\sigma \in \mathbb{R} \setminus \{0\}$ for $\sigma = \sigma_1, \cdots, r$ and

for each $k = \rho_{\sigma_1}, \cdots, n$ we have

$$c_k := \sum_{\substack{\rho_\kappa + \rho_\lambda = 2k \\ \sigma_i \leq k, \lambda \leq \sigma_{i+1}}} (-1)^{\rho_\kappa + k} a_k a_\lambda \geq 0 \tag{6}$$

$(\rho_{\sigma_i} \leq k \leq \rho_{\sigma_i+1}, i = 1, \cdots, s - 1)$

The a_σ ($\sigma = 0, \cdots, \sigma_1 - 1$) may be arbitrary complex constants.

A condition sufficient for (6) to hold is the following simpler condition:

$$\begin{aligned} &\operatorname{sgn}((-1)^{\rho_\sigma/2} a_\sigma) = \text{constant for all } \sigma \geq \sigma_1 \text{ such that } \rho_\sigma \text{ is even} \\ &\operatorname{sgn}((-1)^{(\rho_{\sigma}+1)/2} a_\sigma) = \text{constant for all } \sigma \geq \sigma_1 \text{ such that } \rho_\sigma \text{ is odd} \end{aligned} \tag{7}$$

Definition 2.1

Differential equations of the form (1), satisfying (2), (3), and (6) is called a *special expression of order n* defined on $[1, \infty)$

The expression given by

$$\mathbf{M}_{0,0} f = \sum_{\sigma=0}^{\sigma_1} a_\sigma t^{\alpha_\sigma} f^{(\rho_\sigma)}$$

is called the *essential part* of the special expression $\mathbf{M}_0$.

If we plot the points $(\rho_\sigma, \alpha_\sigma) \in \mathbb{R}^2 (\sigma = 0, \cdots, r)$ in the cartesian plane $\mathbb{R}^2$, and connect $(\rho_\sigma, \alpha_\sigma)$ and $(\rho_{\sigma+1}, \alpha_{\sigma+1})$ $(\sigma = 0, \cdots, r-1)$ by a line segment, then we will obtain a polygonal path with kink (corner) points $(\rho_{\sigma_i}, \alpha_{\sigma_i})$ $(i = 0, \cdots, s-1)$. We will call the σ_i's the *kink indices*, and the polygonal path constructed, the *polygonal path generated by* $\mathbf{M}_0$.

The polygonal path generated by $\mathbf{M}_{0,0}$ lies on or below the bisectrix while the polygonal path generated by $\mathbf{M}_0 - \mathbf{M}_{0,0}$ lies below the bisectrix.

Definition 2.2

Define the function $\gamma : [0, n] \to \mathbb{R}$ by

$$\gamma(l) := \frac{2}{\rho_{\sigma_{i+1}} - \rho_{\sigma_i}} \{ (l - \rho_{\sigma_i}) \alpha_{\sigma_{i+1}} + (\rho_{\sigma_{i+1}} - l) \alpha_{\sigma_i} \}$$

where $\rho_{\sigma_i} \le l \le \rho_{\sigma_{i+1}}$.

Then the graph of γ is precisely the *polygonal path generated by* $\mathbf{M}_0$.

For the case $\alpha_1 > \rho_1$, if we consider expressions of the form (1) where the polygonal path lies above the bisector, things become different. We have a situation where $\mathbf{M}_0$ does not satisfy (2) and thus is no longer a special expression in the same sense. We could however associate a special expression with it.

Definition 2.3

Expression $\mathbf{M}_0$ as in (1) satisfying (3) and (6) that are no longer special expressions, can be associated with a special expression defined to be

$$\mathbf{M}_{0,S} f = t^{-\beta} \mathbf{M}_0 f = t^{\rho_\tau - \alpha_\tau} \mathbf{M}_0 f, \tag{8}$$

where

$$\beta = \max \{ \alpha_i - \rho_i \mid i = 0, \cdots, r \} > 0, \qquad T = \max \{ i \mid \alpha_i - \rho_i = \beta, \alpha_i \neq 0 \}, \tag{9}$$

and

$$\tau = \max \{ i \mid i \in T \}. \tag{10}$$

PERTURBATIONS OF SPECIAL EXPRESSIONS

The perturbation given in this paper was derived from a lemma in Goldberg [2] specialized in the $\mathcal{L}_2$–Hilbert space setting.

Lemma 3.1

Given the interval $I = [1, \infty)$ *and* $\in > 0$, *there exists a* $\tilde{K}$, *depending only on* $\in$, *such that for all* r_l locally in $\mathcal{L}_2(I)$ *and for all* f *in the domain of the maximal operator* $D^{(l+1)}$ $(D = \frac{d}{dt}, l \in \mathbb{N}_0)$

$$\|r_l f^{(l)}\|^2_{2,I} \le (\in \|f^{(l+1)}\|^2_{2,I} + \tilde{K} \|f^{(l)}\|^2_{2,I}) \sup_{[x, x+1] \subset I} \int_x^{x+1} |r_l(t)|^2 dt \tag{11}$$

Proof

Take a compact sub-interval $I_0 = [1, \beta]$ on I.

Let I_1 and I_2 be non-overlapping sub-intervals of I_0 such that $I_1 \cup I_2 = I_0$ with I_1 "to the left" of I_2.

For $\eta > 0$ such that for all $t \in I_1$: $t + \eta \in I_0$ and for all $t \in I_2$: $t - \eta \in I_0$, choose $\varphi \in C^{l+1}([0, \eta])$ so that $0 \le \varphi(t) \le 1$ on $[0, \eta]$, $\varphi(0) = 1$ and $\varphi(\eta) = 0$.

For $f^{(l)} \in$ dom D and t $\in I_1$, we see that

$$f^{(l)}(t) = -\int_0^\eta \frac{d}{dx} (\varphi(x) f^{(l)}(t+x))\, dx$$
$$= -\int_0^\eta \varphi(x) f^{(l+1)}(t+x))\, dx - \int_0^\eta \varphi'(x) f^{(l)}(t+x))\, dx.$$

Letting $M = \max_{0\le x\le\eta} |\varphi'(x)|$, we obtain

$$|f^{(l)}(t)| \le \int_0^\eta |f^{(l+1)}(t+x)|\,dx + M\int_0^\eta |f^{(l)}(t+x))|\,dx$$

$$\le \eta^{1/2}\left[\left(\int_0^\eta |f^{(l+1)}(t+x)|^2dx\right)^{1/2} + M\left(\int_0^\eta f^{(l)}(t+x)|^2dx\right)^{1/2}\right]$$

It follows from the Schwarz's inequality in $1R^2$ that

$$|f^{(l)}(t)| \le \eta^{1/2}\left[(1,1)\begin{pmatrix}\left(\int_0^\eta |f^{(l+1)}(t+x)|^2dx\right)^{1/2}\\ \left(M^2\int_0^\eta |f^{(l)}(t+x)|^2dx\right)^{1/2}\end{pmatrix}\right]$$

$$\le \eta^{1/2}(2)^{1/2}\left[\left(\int_0^\eta |f^{(l+1)}(t+x)|^2dx\right) + \left(M^2\int_0^\eta |f^{(l)}(t+x)|^2dx\right)\right]^{1/2}$$

$$\le 2\eta^{1/2}\left[\left(\int_0^\eta |f^{(l+1)}(t+x)|^2dx\right) + M^2\left(\int_0^\eta |f^{(l)}(t+x)|^2dx\right)\right]^{1/2} \quad (12)$$

Taking η sufficiently small, it follows that there exists a K_1 depending only on $\in$ such that $t \in I_1$,

$$|f^{(l)}(t)| \le \frac{\in}{2}\int_0^\eta |f^{(l+1)}(t+x)|^2dx + K_1\int_0^\eta |f^{(l)}(t+x)|^2dx$$

$$\le \frac{\in}{2}\int_t^{t+\eta} |f^{(l+1)}(x)|^2dx + K_1\int_t^{t+\eta} |f^{(l)}(x)|^2dx. \quad (13)$$

From the conditions put on η we see that as β approaches infinity, η can be kept fixed so that (12) still holds.

Thus K_1 can be chosen to be a non-increasing function of the length of I_0.

Letting a be the left endpoint of I_1, (13) implies that

$$\int_{I_1} |r_l(t)f^{(l)}(t)|^2\,dt \le \int_{I_1}\int_t^{t+\eta} |r_l(t)|^2\left(\frac{\in}{2}|f^{(l+1)}(x)|^2dx + K_1|f^{(l)}(x)|^2\right)dxdt$$

$$= \int_{I_1}\int_I |r_l(t)|^2\,\Upsilon_{[t,t+\eta]}(x)\left(\frac{\in}{2}|f^{(l+1)}(x)|^2 + K_1\int |f^{(l)}(x)|^2\right)dxdt$$

where $\Upsilon_{[t,t+\eta]}(x) = \begin{cases} 0 \text{ if } t > x \text{ or } t+\eta \le x \\ 1 \text{ if } t \le x < t+\eta \end{cases}$

By Fubini's theorem, we have

$$\int_{I_1} |r_l(t)f^{(l)}(t)|^2\,dt$$

$$\le \int_I\left(\frac{\in}{2}|f^{(l+1)}(x)|^2dx + K_1|f^{(l)}(x)|^2\right)\int_{I_1} |r_l(t)|^2\,\Upsilon_{[t,t+\eta]}(x)\,dtdx$$

$$= \int_I\left(\frac{\in}{2}|f^{(l+1)}(x)|^2 + K_1\int |f^{(l)}(x)|^2\right)\int_{\max(x-\eta,a)}^x |r_l(t)|^2\,dtdx$$

$$\le \int_I\left(\frac{\in}{2}|f^{(l+1)}(x)|^2 + K_1\int |f^{(l)}(x)|^2\right)\int_x^{x+1} |r(\tau-\eta)|^2\,d\tau dx$$

$$= \int_I\left(\frac{\in}{2}|f^{(l+1)}(x)|^2 + K_1\int |f^{(l)}(x)|^2\right)\sup_{[x,x+1]\subset I}\int_x^{x+1} |r_l(t)|^2\,dtdx$$

$$= \frac{\in}{2}\int_I\left(|f^{(l+1)}(x)|^2 + K_1\int |f^{(l)}(x)|^2\right)dx\ \sup_{[x,x+1]\subset I}\int_x^{x+1} |r_l(t)|^2\,dt$$

$$= \left(\frac{\in}{2}\|f^{(l+1)}\|_{2,I}^2 + K_1\|f^{(l)}(x)\|_{2,I}^2\right)dx\ \sup_{[x,x+1]\subset I}\int_x^{x+1} |r_l(t)|\,dt. \quad (14)$$

For $t \in I_2$, we have

$$|f^{(l)}(t)| = -\int_0^\eta \frac{d}{dx}\left(\varphi(x)f^{(l)}(t-x)\right)dx.$$

Thus, by the argument established in (13), for $t \in I_2$, we get

$$|f^{(l)}(t)| \le \frac{\in}{2}\int_{t-\eta}^t |f^{(l+1)}(x)|^2\,dx + K_1\int_{t-\eta}^t |f^{(l)}(x)|^2\,dx.$$

As in (13), we obtain

$$\int_{I_2} |r_l(t) f^{(l)}(t)|^2\, dt$$

$$\leq \int_I \left(\frac{\epsilon}{2} |f^{(l+1)}(x)|^2 + K_1 |f^{(l)}(x)|^2\right) dx \int_x^{x+\eta} |r_l(t)|^2\, dt$$

$$\leq \left(\frac{\epsilon}{2} \| f^{(l+1)} \|_{2,I}^2 + K_1 \| f^{(l)}(x) \|_{2,I}^2\right) \sup_{[x,\, x+1] \subset I} \int_x^{x+1} r_l(t)\, dt. \tag{15}$$

The lemma thus follows from (14) and (15).

Let us now define a relatively compact perturbation of special expressions.

Definition 3.2

Let **M** be a differential expression of the form

$$\mathbf{M} f = \sum_{l=0}^{n-1} r_l(t) f^{(l)} \quad . \tag{16}$$

We say that **M** is an *admissible perturbation* of the differential expression $\mathbf{M}_0$ if there exists a $\boldsymbol{B}$ such that the coefficients r_l satisfy the following:

$$\sup_{[x,\, x+1] \subset I} \int_x^{x+1} \left| \frac{r_l(t)}{s_l(t)} \right|^2 dt < \boldsymbol{B} \tag{17}$$

where $r_l \in \mathbf{C}^l(I)$ for $l = 0, \cdots, n-1$ and $0 < s_l$ is an auxilliary function $s_l \in C^\infty(I)$ satisfying

$$s_l(t) = o\left(t^{\frac{1}{2}\gamma(l+1)}\right) \text{ and } s_l(t) = o\left(t^{\frac{1}{2}\gamma(l)}\right) \tag{18}$$

For the invariance of nullities, we can only admit a somewhat less general class of perturbations consisting of expressions (16) satisfying:

$$\sup_{[x,x+1] \subset I} \int_x^{x+1} \left| \frac{r_l^{(l-b)}(t)}{s_b(t)} \right|^2 dt < \tilde{B} \tag{19}$$

for $b = 0, \cdots, n-l$, $l = b, \cdots, n-l$.

Definition 3.3

Let **M** be a differential expression of the form (16). Take $\tilde{\gamma}(l) = \min\{\gamma(l+1), \gamma(l)\}$. We call **M** an admissible perturbation of the form (1) with $\alpha_1 > \rho_1$, if

$$s_l(t) = \begin{cases} o\left(t^{\frac{1}{2}\tilde{\gamma}(l)}\right), \text{ if } = \frac{\tau}{\sigma_1}, \cdots, s-1 \text{ exists with } \rho_{\sigma_i} \leq l \leq \rho_{\sigma_{i+1}} \\ o\left(t^{(\beta+l)}\right), \text{ for } l = 0, \cdots, \rho_\tau \end{cases} \tag{20}$$

or respectively for $b = 0, \cdots, n-1$, $l = b, \cdots, n-1$.

$$\sup_{[x,x+1] \subset I} \int_x^{x+1} \left| \frac{(t^{-\beta} r_l)^{(l-b)}(t)}{t^{-\beta} s_b(t)} \right|^2 dt < \tilde{B} \tag{21}$$

The following proposition of Schultze [5] will be used to prove the main result.

Proposition 3.4

If $\mathbf{M}_0$ *is a special expression, then there are constants* $b_l > 0$ $(l = 0, \cdots, n)$, $K \geq 0$, and $\eta \in I$ such that *for all* $f \in C_0^\infty(\eta, \infty)$, we have

$$\| \mathbf{M}_0 f \|_2^2 \geq \sum_{l=0}^{n} \int_I b_l t^{\gamma(l)} |f^{(l)}|^2 + (b_0 - K) \| f \|_2^2 \tag{22}$$

The theorem below that is due to Kauffman is the basic perturbation theorem that we will apply.

Theorem 3.5

Let $\mathbf{M}_0$ *be given as in* (1) *and* $\mathcal{R}(\mathbf{T}_0(\mathbf{M}))$ *closed. Let* **M** *be another expression of the form* (16) *with order* **M** < *order* $\mathbf{M}_0$ *satisfying the following condition:*

There is a $g \in C([1,\infty])$, $g > 0$, *and* $\lim_{t\to\infty} g(t) = \infty$ *such that* $g\mathbf{M}f \in \mathcal{L}_2([1,\infty])$ for all $f \in \mathcal{D}(\mathbf{T}_0(\mathbf{M}_0))$. *Then the operator, defined as the restriction of* **M** *on* $\mathcal{D}(\mathbf{T}_0(\mathbf{M}_0))$, *is relatively compact with respect to* $\mathbf{T}_0(\mathbf{M}_0)$, *and we have*

$$\mathcal{D}(T_0(M_0 + M)) = \mathcal{D}(T_0(M_0)),$$
$$\text{nul}(M^+ + N^+) = \text{nul}\, M^+$$
$$\mathcal{R}(T_0(M_0 + M)) \text{ is closed.} \tag{22}$$

With the following lemma we can show that

$$\mathcal{D}(T_0(M_0 + M)) = \mathcal{D}(T_0(M_0))$$

Lemma 3.6

Let M_0 *be a special expression and* M *a corresponding expression, i.e., of the form* (16) *satisfying* (17) *and* (18). *Then there exist* $\eta \in I$, $0 < \alpha < 1$, $\beta \geq 0$ *such that for all* $f \in C_o^\infty(\eta, \infty)$, *we have*

$$\|Mf\|_2 \leq \alpha \|M_0 f\|_2 + \beta \|f\|_2 \tag{23}$$

MAIN RESULTS

The crucial lemma that will show the main result is the following:

Lemma 4.1

There exists $0 < g \in C([1,\infty])$ *with* $\lim_{t\to\infty} g(t) = \infty$, *and constants* $\eta \in I$, $0 < \alpha < 1$, $\beta \geq 0$ *such that for* $f \in C_0^\infty(\eta, \infty)$ *and* $l = 0, \cdots, n-1$ *we have*

$$\| r_l g f^{(l)} \|_2 \leq \alpha \| M_0 f \|_2 + \beta \| f \|_2.$$

Theorems 4.2 and 4.3 give the invariance of the essential spectrum and the nullities under the relatively compact perturbation we have obtained.

Theorem 4.2

Let M_0 *be a special expression and* M *a corresponding perturbation, i.e., an expression of the form* (16) *satisfying* (17) *and* (18). *Then*

$$\sigma_e(M_0 + M) = \sigma_e(M_0) = \sigma_e(M_{0,0}),$$

where

$$\sigma_e(M_{0,0}) = \begin{cases} \{ \sum_{\sigma=0}^{\sigma_1} a_\sigma z^{\rho_\sigma} \mid \text{Re}\, z = 0\}, & \text{for } \alpha_1 < \rho_1 \\ \{ \sum_{\sigma=0}^{\sigma_1} a_\sigma \prod_{j=0}^{\sigma-1} (z - \frac{1}{2} - j) \mid \text{Re}\, z = 0\}, & \text{for } \alpha_1 = \rho_1 \end{cases}$$

In addition, if M *satisfies* (19), *then for every* $\lambda \in \mathbb{C} \setminus \sigma_e(M_0)$,

$$\text{nul}(M_0 + M - \lambda) = \text{nul}(M_0 - \lambda)$$

$$= \text{nul}(M_{0,0} - \lambda) + \sum_{i=1}^{s-1} \#\{z \mid \sum_{\sigma=\sigma_i}^{\sigma_i + 1} a_\sigma z^{\rho_\sigma} = 0, \text{Re}\, z < 0\}$$

where

$$\text{nul}(M_{0,0} - \lambda) = \begin{cases} \#\{z \mid \sum_{\sigma=0}^{\sigma_1} a_\sigma z^{\rho_\sigma} = \lambda, \text{Re}\, z < 0\}, & \text{for } \alpha_1 < \rho_1 \\ \#\{z \mid \sum_{\sigma=0}^{\sigma_1} a_\sigma \prod_{j=0}^{\sigma-1} (z - ½ - j) = \lambda, \text{Re}\, z < 0\}, & \text{for } \alpha_1 = \rho_1 \end{cases}$$

Outline of the Proof

From Lemma 4.1 and Theorem 3.5 we obtain, for closed $\mathcal{R}(T_0(M_0 - \lambda))$ we have $\mathcal{R}(T_0(M_0 + M - \lambda))$ is closed and $\text{nul}((M_0 - \lambda)^+) = \text{nul}((M_0 + M - \lambda)^+)$.

And if $\mathcal{R}(T_0(M_0 + M - \lambda))$ is closed, it follows in the same way that $\mathcal{R}(T_0(M_0 - \lambda))$ is closed. Hence we have proven the first assertion, that is

$$\sigma_e(M_0 + M) = \sigma_e(M_0).$$

However for the invariance of the nullities, what we have shown is that

$$\text{nul}(M_0 - \lambda)^+ = \text{nul}(M_0 + M - \lambda)^+,$$

then to complete the proof of the theorem we must show that M^+ is a relatively compact perturbation of a certain special expression.

This can be done by giving appropriate definitions for M_0^+ and $\tilde{M}_0^+$ such that $\tilde{M}_0^+$ is a special expression with the same polygonal path (same $\gamma(l)$ as M_0).

Theorem 4.3

Let $\mathbf{M}_0$ *be of the form* (1) *satisfying* (9), (3) *and* (6), *let* $\mathbf{M}_{0,s}$ *be given by* (8) *and let* $\mathbf{M}$ *be given by* (16) *satisfying* (17) *with* (20). *Then if* $0 \in C \setminus \sigma_e(\mathbf{M}_{0,s})$,

$$\sigma_e(\mathbf{M}_0 + \mathbf{M}) = \varnothing$$

and if $\mathbf{M}$ *satisfies even* (21), *we have for all* $\lambda \in \mathbb{C}$,

$$\text{nul}\,(\mathbf{M}_0 + \mathbf{M} - \lambda) = \text{nul}\,(\mathbf{M}_{0,s}).$$

Proof

Thoerem 4.2 applied on $\mathbf{M}_{0,s}$ and $t^{-\beta}(\mathbf{M} - \lambda)$ give for $\lambda \notin C$

$$\sigma_e(M_{0,s}) = \sigma_e(M_{0,s} + t^{-\beta}(M - \lambda))$$

If $0 \in \mathbb{C}\setminus\sigma_e(\mathbf{M}_{0,s})$ and since I contains one of its end-points, then there exists $K > 0$ such that for all $f \in \mathcal{D}(\mathbf{T}_0(\mathbf{M}_0 + \mathbf{M}))$ we have

$$\|(\mathbf{M}_0 + \mathbf{M} - \lambda)f\|_2 \geq K\|f\|_2$$

Therefore, $\lambda \in \mathbf{C} \setminus \sigma_e(M_0 + M)$.

And if M satisfies (21), it follows from Theorem 4.2 that

$$\text{nul}\,(\mathbf{M}_{0,s}) = \text{nul}\,(\mathbf{M}_{0,s} + t^{-\beta}(\mathbf{M} - \lambda) = \text{nul}\,(\mathbf{M}_0 + \mathbf{M} - \lambda).$$

The perturbation given by Definition 3.2 was also applied on self-adjoint differential expression on $I = [a, \infty)$, $a \in$ 1R where the coefficient of the highest derivative has no zeros on I. This class of expressions resulted to a spectrum that is maximal. By this we mean that the essential spectrum is the whole set of real numbers.

The following theorems give our results.

Theorem 4.4

Let

$$Mf := \sum_{\mu=0}^{\sigma} a_\mu t^{\beta\cdot\mu} f^{(\mu)} + \sum_{l=0}^{m} b_l(t) f^{(l)} \tag{24}$$

with $\beta \leq 1, 0 < \sigma$, $a_\sigma \neq 0$, $a_\mu \in \mathbb{C}$ $(\mu = 0, \cdots, \sigma)$, $b_l \in C^l(I, \mathbb{C})$ $(l = 0, \cdots, m-1)$ *such that*

$$\sup_{[x,x+1]\subset I} \int_x^{x+1} \left|\frac{b_l(t)}{s_l(t)}\right|^2 dt < B$$

where $0 < s_l$ is an auxilliary function $s_l \in C^\infty(I)$ satisfying $s_l(t) = o(t^{l.b})$ and $s_l(t) = o(t^{(l+1).\beta})$ for $l = 0, \cdots, \sigma$ and with $\gamma < \beta$: $s_l(t) = O(t^{l\cdot\gamma})$ and $s_l(t) = O(t^{(l+1).\gamma})$ for $l = \sigma + 1, \cdots, m$.

Then

$$\sigma_e(\mathrm{M}) \supset \begin{cases} \{\sum_{\mu=0}^{\sigma} a_\mu z^\mu \mid \operatorname{Re} z = 0\}, & \text{for } \beta < 1 \\ \{\sum_{\mu=0}^{\sigma} a_\mu \prod_{j=0}^{\mu-1} (z - \tfrac{1}{2} - j) \mid \operatorname{Re} z = 0\}, & \text{for } \beta = 1 \end{cases} \tag{25}$$

Outline of the Proof

Define

$$\mathbf{N}_0 f := \sum_{\mu=0}^{\sigma} a_\mu t^{\beta\cdot\mu} f^{(\mu)}$$
$$\mathbf{N}_1 f := \sum_{\mu=0}^{\sigma} a_\mu t^{\beta\cdot\mu} f^{(\mu)} + t^{\frac{n}{2}(\beta+\gamma)} f^{(n)}$$

$$\text{where } n := \begin{cases} \mathrm{m}+1 \text{ if } m - \sigma \text{ is even} \\ \mathrm{m}+2 \text{ if } m - \sigma \text{ is odd} \end{cases}$$

Then $\mathbf{N}_0$ and $\mathbf{N}_1$ are special expressions of the form (1) with essential part $\mathbf{N}_0$ and $\mathbf{N}_1$ having only one kink index σ, we have

$$\|(\mathrm{N}_1 - \lambda)f\|_2^2 \geq \sum_{l=0}^{\sigma} \int_I \mathrm{C}_l t^{\gamma(l)} |f^{(l)}|^2 + \sum_{l=\sigma+1}^{n} \int_I \mathrm{C}_l t^{\gamma(l)} |f^{(l)}|^2 - \mathrm{K}\|f\|_2^2 \tag{26}$$

where

$$\gamma(l) := \begin{cases} 2\beta.1 & \text{for } 0 \leq l \leq \sigma \\ \frac{2}{n-\sigma}\{(l-\sigma)\frac{n}{2}(\beta+\gamma) + (n-l)\beta\sigma\} & \text{for } \sigma < l \leq n. \end{cases}$$

With

$$N_2 f := t^{\frac{n}{2}(\beta+\lambda)} f^{(n)}$$

we obtain the following estimate:

$$\|(N_0-\lambda)f\|_2^2 \le (1+\frac{3\alpha_1}{c})\|(N_1-\lambda)f\|_2^2 + \frac{3\alpha_1}{c}\|f\|_2^2 \quad (27)$$

Let

$$M_1 f = \sum_{l=0}^{m} b_l(t) f^{(l)}$$

$$\|M_1 f\|_2^2 \le \alpha_2 \|N_1 f - \lambda\|_2^2 + \alpha_2 K \|f\|_2^2 \quad (28)$$

we see that there exists $G_\lambda > 0$ and $\eta \in I$ such that for all $f \in C_0^\infty(\eta,\infty)$

$$\|(N_1-\lambda)f\|_2^2 \ge G_\lambda \|f\|_2^2$$

But this implies that $\lambda \notin \sigma_e(N_1)$ and from Theorem 4.2 follows that λ is not contained in the right hand side of (25). Since this holds for arbitrary λ, we have shown the assertion.

Theorems 4.5 and 4.6 are consequences of Theorem 4.4.

Theorem 4.5

Let

$$M_0 f := \sum_{\mu=0}^{k} a_{2\mu}(t^{2\beta\mu} f^{(\mu)})^{(\mu)} +$$

$$\frac{i}{2}\sum_{\mu=0}^{k} a_{2\mu+1}\{(t^{\beta(2\mu+1)} f^{(\mu)})^{(\mu+1)} + (t^{\beta(2\mu+1)} f^{(\mu+1)})^{(\mu)}\}$$

with $\beta \le 1$, $a_l \in \mathbb{R}(l = 0, \cdots, 2k+1)$, $a_{2k+1} \ne 0$ *and let*

$$Nf = \sum_{l=0}^{m} r_l(t) f^{(l)}$$

be self adjoint with $r_l \in C^l \ (I, \mathbb{C})$ *such that*

$$\sup_{[x,x+1]\subset I} \int_x^{x+1} \left|\frac{r_l(t)}{s_l(t)}\right|^2 dt < \infty$$

where $0 < s_l$ *is an auxilliary function* $s_l \in C^\infty(I)$ *satisfying for some* $\gamma < \beta$:

$s_l(t) = o(t^{l\cdot\beta})$ *and* $s_l(t) = o(t^{(l+1)\cdot\beta})$ *for* $l = 0, \cdots, 2k+1$
$s_l(t) = O(t^{l\cdot\gamma})$ *and* $s_l(t) = O(t^{(l+1)\cdot\gamma})$ *for* $l = 2k+2, \cdots, m$.

Then

$$\sigma_e(M_0 + N) = \mathbb{R}.$$

Proof

Since $(M_0 + N)$ is self adjoint, then $\sigma_e(M_0 + N) \subset \mathbb{R}$. By Theorem 4.4, this essential spectrum contains the range of an odd-order polynomial which is all of $\mathbb{R}$.

Theorem 4.6

Let

$$M_0 f := \sum_{\mu=0}^{k} (-1)^\mu a_{2\mu}(t^{2\beta\mu} f^{(\mu)})^{(\mu)} +$$

$$\frac{i}{2}\sum_{\mu=0}^{k-1} (-1)^{\mu+1} a_{2\mu+1}\{(t^{\beta(2\mu+1)} f^{(\mu)})^{(\mu+1)} + (t^{\beta(2\mu+1)} f^{(\mu+1)})^{(\mu)}\}$$

with $\beta \le 1$, $a_l \in \mathbb{R}$ $(l = 0, \cdots, 2k)$, $a_{2k} > 0$ *and let*

$$Nf = \sum_{l=0}^{m} r_l(t) f^{(l)}$$

be self adjoint with $r_l \in C^l \ (I, \mathbb{C})$ *and*

$$\sup_{[x,x+1]\subset I} \int_x^{x+1} \left|\frac{r_l(t)}{s_l(t)}\right|^2 dt < \infty$$

where $0 < s_l$ *is an auxilliary function* $s_l \in C^\infty(I)$ *satisfying with some* $\gamma < \beta$:

$s_l(t) = o(t^{l\cdot b})$ *and* $s_l(t) = o(t^{(l+1)\cdot\beta})$ *for* $l = 0, \cdots, 2k$
$s_l(t) = O(t^{l\cdot\gamma})$ *and* $s_l(t) = O(t^{(l+1)\cdot\gamma})$ *for* $l = 2k+2, \cdots, m$.

Also let

$$\Lambda := \begin{cases} \min\{\sum_{\tau=0}^{2k} \alpha_\tau x^\tau \mid x \in \mathbb{R}\}, & \text{for } \beta<1 \\ \min\{\sum_{\tau=0}^{2k} a_\tau x^{\tau-2[\frac{\tau}{2}]} \prod_{\xi=0}^{[\frac{\tau}{2}-1]} (x^2 + (\frac{2\xi+1}{2})^2) \mid x\in\mathbb{R}\}, & \text{for } \beta=1. \end{cases}$$

Then

$\sigma_e(\mathbf{M}_0+\mathbf{N}) \supset [\Lambda, \infty)$.

Proof

For $\beta < 1$, differentiating $\mathbf{M}_0$ will give terms involving derivatives of the coefficients falling under the o-terms. The first case of Theorem 4.4 proves the assertion with $\sigma = 2k$, $z = ix$, and $x \in$ 1R.

For $\beta = 1$, the derivatives of the coefficients have to be taken into account. The following observation is helpful. If

$$\mathrm{M}f = \sum_{\mu=0}^{2k} \tilde{a}_{\mu} t^{\mu} f^{(\mu)} \quad \text{then}$$

$$\mathrm{M}t^{ix-\frac{1}{2}}\Big|_{t=1} = \sum_{\mu=0}^{2k} \tilde{a}_{\mu} \prod_{j=0}^{\mu-1} \left(ix - \tfrac{1}{2} - j\right).$$

Therefore the polynomial on the right side of (25) is

$$\mathbf{M}_0 t^{ix-\frac{1}{2}}\Big|_{t=1} = \sum_{\mu=0}^{k} a_{2\mu} \prod_{j=0}^{\mu-1} \left(x^2 + (j+\tfrac{1}{2})^2\right) + \prod_{j=0}^{k-1} a_{2\mu+1} \prod_{j=0}^{\mu-1} \left(x^2 + (j+\tfrac{1}{2})^2\right).$$

which is the polynomial that appears in Λ for $\beta = 1$

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