An Existence Theorem for Differential Inclusions Using the Kurzweil Integral

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ABSTRACT

We define solutions to the differential inclusion $x(t) \in F(t,x)$ in the Kurzweil sense, using the Kurzweil integral of a multifunction, and establish an existence theorem for these inclusions under a semicontinuity condition called Property (Q), and the assumption of integrable-boundedness.

INTRODUCTION

Let $F(t,x)$ be a nonempty subset of Euclidean n-space $\Re^n$. The initial value problem

$$\dot{x} \in F(t,x(t)), \qquad x(0) = x_o \tag{1}$$

is a generalization of the initial value problem in ordinary differential equations. A solution of inclusion (Eqn. 1) is defined as an absolutely continuous function ϕ, with $\phi(0) = x_0$, and $\dot{\phi}(t) \in F(t,\phi(t))$ for almost every t in a neighborhood of 0.

Inclusions of the form (Eqn. 1) arise in several ways, perhaps the most familiar one being in the theory of control systems, when dealing with equations of the form

$$x = f(t,x,u), \qquad x(0) = x_0 \tag{2}$$

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where the control parameter u may be chosen as any measurable vector-valued function with $u(t) \in U(t,x) \subset \Re^n$. In this case, the right-hand side of the differential inclusion is

$$F(t,x) = \{ f(t,x,u) \mid u \in U(t,x) \}. \tag{3}$$

Differential inclusions may also arise, for example, from the consideration of implicit differential equations $f(t, x, \dot{x}) = 0$. Finally, differential inequalities may also be recast as differential inclusions (Hermes, 1970).

SET-VALUED FUNCTIONS AND DEFINITIONS

We denote the Euclidean distance between two points x, y in $\Re^n$ as $|x - y|$ or $\rho(x,y)$. If A is a subset of $\Re^n$, the distance between the point x and A is

$$\rho(x,A) = \inf \{\rho(x,a): a \in A \}. \tag{4}$$

For any $A \subset \Re^n$, the closed neighborhood $V(A,\varepsilon)$ of the A is defined as

$$V(A,\varepsilon) = \{ x \in \Re^n \mid \rho(x,A) \leq \varepsilon\}. \tag{5}$$

The Hausdorff distance between two sets A, B in $\Re^n$, denoted by $h(A,B)$ is defined as

$$h(A,B) = \inf\{ \varepsilon > 0 \mid A \subset V(B,\varepsilon) \text{ and } B \subset V(A,\varepsilon)\}. \quad (6)$$

The collection of all nonempty compact subsets of $\Re^n$ with the topology induced by the Hausdorff distance is a complete metric space, and we denote it by Ω^n.

Convexity will be important in what follows. We recall the definition and emphasize that it is a concept independent of the topological structure.

Definition 2.1 *A subset of the linear space X is said to be* convex *if whenever it contains* x_1 *and* x_2, *it also contains* $\lambda_1 x_1 + \lambda_1 x_2$, where $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, *and* $\lambda_1 + \lambda_2 = 1$.

Definition 2.2 *Given a subset A of a linear space X, the* convex hull *of A, denoted co(A), is the smallest convex set that contains A. (It is also the intersection of all convex sets that contain A.)*

We will use Conv($\Re^n$) to denote the subset of Ω^n whose elements are convex sets.

Measurability is also an important property for set-valued functions.

Definition 2.3 *Let F(t) be a set-valued function defined on a real interval I, with values in a separable metric space Y. F is said to be* measurable *(in the sense of Lebesque) if, for every closed subset* $D \subset Y$, *the set* $\{t \in I : F(t) \cap D \neq \emptyset\}$ *is Lebesque-measurable in* $\Re^1$.

The notion of continuity for set-valued functions is formulated in various ways. We will deal with the following formulations.

Definition 2.4 *F is said to be* upper semicontinuous *(u.s.c.) at* $x_0 \in X$ *if for any open N containing* $F(x_0)$, *there exists a neighborhood M of* x_0 such that $F(M) \subset N$. *F is upper semicontinuous if it is u.s.c. at every* $x_0 \in X$. *(Equivalently, for each closed* $A \subset Y$, $X \setminus F^{-1}(2^{Y \setminus A}) = \{x: F(x) \cap A \neq \emptyset\}$ *is closed in X* (Kuratowski, 1966).

Definition 2.5 (Property K) *Suppose X is a metric space, Y is a linear topological space, and F(x) is a set-valued map from X to Y. Let* $x_0 \in X$. *F is said to satisfy Property (K), (Kuratowski's concept of upper semicontinuity) at* x_0 *provided* $F(x_0) = \cap_\delta\, cl\, F(x_0, \delta)$, that is,

$$F(x_0) = \bigcap_{\delta > 0} cl \bigcup_{x \in N_\delta(x_0)} F(x). \quad (7)$$

Here, cl refers to 'closure.' F(x) has Property (K) if it does, at every $x_0 \in X$.

The corresponding definition, suitable for convex sets, was introduced by Cesari (1983).

Definition 2.6 Property (Q). *F:* $x \to F(x)$ *has Property (Q) at* x_0 *provided*

$$F(x_0) = \bigcap_{\delta > 0} cl\, co \bigcup_{x \in N_\delta(x_0)} F(x). \quad (8)$$

Here, cl co refers to 'closed convex hull.' F(x) has Property (Q) if it does, at every $x_0 \in X$.

For a discussion on relationships among Properties (K), (Q) and upper-semicontinuity, see Cesari (1983).

The following theorem does not deal with set-valued maps. However, it is crucial in proving results on existence of solutions to differential inclusions, under appropriate conditions. The theorem is due to Aumann (1965); a shorter proof may be found in Davy (1972).

Theorem 2.1 *Let* $\{x_k\}$ *be a sequence of absolutely continuous functions* $x_k : I \to \Re^n$. *Suppose that* $x_k(t) \to x(t)$ *as* $k \to \infty$, *for all* $t \in I$, *where* $x : I \to \Re^n$, and $|\dot{x}_k(t)| \leq g(t)$ *a.e.* $t \in I$, *where* $g : I \to \Re$ *is a Lebesque-integrable function. Then, x is an absolutely continuous function such that*

$$\dot{x}(t) \in \bigcap_{i=1}^{\infty} cl\, co \bigcup_{k=i}^{\infty} \dot{x}_k(t) \quad (9)$$

for a.e. $t \in I$.

The inclusion may be compared with the condition for a multifunction to satisfy Property (Q). As will be seen in the sequel, Theorem 2.1, together with Property (Q) (or upper semicontinuity), will play a crucial role in proving existence theorems.

APPROACHES TO THE EXISTENCE PROBLEM FOR DIFFERENTIAL INCLUSIONS

As pointed out by Aubin and Cellina (1984), for differential inclusions, the condition to be imposed on the set-valued mapping F in order to ensure existence of solutions are of two kinds: regularity conditions on the map (the various kinds of continuity or semicontinuity, for instance), and conditions of topological or geometric type (compactness, convexity) on the images of points. Various combinations are possible, and one case is considered by Davy (1972). His assumptions are that F maps from $[a, b] \times \Re^n$ into Ω^n ; $F(t,x)$ is convex; the mapping $x \to F(t,x)$ is upper semicontinuous on $\Re^n$ for all $t \in I$; for all $x \in \Re^n$, there exists $f_x : I \to \Re^n$ such that f_x is measurable and $f_x(t) \in F(t,x)$; and there exists $g \in L^1(I)$ such that $y \in F(t,x)$ implies that $|y| \leq g(t)$. With these assumptions, the following existence theorem is proved:

Theorem 3.1 *There exist solutions to the inclusion*

$$\dot{x} \in F(t,x(t)), \qquad x(0) = x_0. \tag{10}$$

Trajectories are also shown to satisfy properties proven for solutions of ordinary differential equations, namely: compactness, Kneser's and Hukuhara's property.

In the case considered by Davy (1972), although no reference is made to an integral inclusion formulation of the differential inclusion, it is the case that, with the integral of a set-valued function appropriately defined, the differential inclusion is equivalent to an integral inclusion. The integral used in the inclusion is an Aumann integral, defined as follows:

Definition 3.1 (The Aumann Integral of a Set-Valued Map) *Let I be some closed interval $[a,b]$. For each $t \in I$, let $F(t)$ be a non-empty subset of Euclidean space $\Re^n$. Let ϑ be the set of all point-valued functions f from I to $\Re^n$ such that f is Lebesgue-integrable over I and $f(t) \in F(t)$ for all $t \in I$. Define*

$$\int_I F(t)\, dt := \{ \textstyle\int_I f(t)\, dt \mid f \in \vartheta \}, \tag{11}$$

the set of all integrals of members of ϑ. This integral will also be denoted by $(A)\int_I F(t)\, dt$.

The Aumann integral is the most well-known integral considered in the theory of integration of set-valued maps. Clearly, the question of the nonemptiness of the integral depends on the existence of selections, that is, of single-valued functions each of whose values belong to the corresponding set, i.e., $f(t) \in F(t)$. There is a long literature concerned with such selection problems (Aubin and Cellina, 1992; Deimling, 1992; Castaing and Valadier, 1970; Wagner, 1977) In this paper, we will rely on a theorem of Kuratowski and Ryll-Nardzewski, which can be stated as follows:

Theorem 3.2 (Kuratowski and Ryll-Nardzewski). *If (X, ρ) is a separable complete metric space and $F:T \to \wp(X)$ is measurable and has closed values, then F has a measurable selector.*

For a given set-valued map F on and interval I, F is said to be integrably bounded if there exists a single-valued Lebesgue-integrable function $g(t)$ such that for $t \in I$, $|y| \leq g(t)$ *whenever $y \in F(t)$. If F is measurable and integrably bounded, then $\int F$ is non-empty.*

For Aumann integrals, the equivalence of the differential and integral inclusion is well-known. For example, in Aubin and Cellina (1984), the following lemma is proven (Castaing and Valadier, 1970):

Lemma 3.3 (Integral Representation) *Let F be an upper-semicontinuous map from $I \times \Re^n$ into the compact convex subsets of $\Re^n$. Then the continuous function x is a solution on I to the inclusion*

$$\dot{x}(t) \in F(t, x(t)) \tag{12}$$

if and only if for every pair t_1 and $t_2 \in I$,

$$x(t_2) \in x(t_1) + \int_{t1}^{t2} F(s,x(s))\, ds. \tag{13}$$

While the Aumann integral seems to be sufficient in many instances when integrals of set-valued functions are used, a natural question that arises, from the Kurzweil-Henstock generalization of the Lebesque integral of a point-valued function, is whether a similar generalization can be made, which includes the Aumann integral. Such a generalization was in fact initiated by Artstein and Burns (1975), and their construction was further modified by Jarnik and Kurzweil (1983). In the next section, we will outline their generalization, which, as we will see, follows the Riemann-sum type of construction made in the single-valued case. In view of that generalization, one can raise the following question:

If we allow all solutions to the integral inclusion

$$x(t) \in x(t_o) + \int_{t_0}^{t} F(s,x(s))\, ds \qquad \textbf{(14)}$$

(with the integral on the right interpreted as the Kurzweil integral), to qualify as solutions to the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \qquad \textbf{(15)}$$

in some interval containing t_0, and if we in fact define the differential inclusion (Eqn. 15) to mean integral inclusion (Eqn. 14) (i.e., their solution sets coincide), are we broadening the class of solutions?

We will investigate the 'new' differential inclusion, after we describe the integral of a set-valued mapping constructed by Jarnik and Kurzweil (1983).

THE JARNIK-KURZWEIL INTEGRAL OF A SET-VALUED MAPPING

A *partition* of the interval $I := [a,b]$ is a collection $\Delta = \{ (t_j, [a_{j-1}, a_j]), j = 1, \dots k;\ a = a_0 \leq a_1 \leq \dots \leq a_k = b\}$, and a *gauge* on I is defined as a positive real-valued function $\delta : I \to (0, \infty)$. Δ is *subordinate* to δ (briefly: Δ sub δ) if

$$[a_{j-1}, a_j] \subset (t_j - \delta(t_j), t_j + \delta(t_j)), j = 1, \dots, k. \qquad \textbf{(16)}$$

The *Riemann sum* for F corresponding to a partition D is

$$S(F, \Delta) := \{\textstyle\sum_{j=1}^{k} \phi_j (a_j - a_{j-1}) \mid \phi_j \in F(t_j), j = 1, \dots, k\}. \qquad \textbf{(17)}$$

For brevity, one writes $S(F, \Delta) := \sum_{j=1}^{k} F(t_j)(a_j - a_{j-1})$, using the usual definition of the sum and multiple of sets.

For a multifunction F, denote by $\Theta(F)$ the multifunction with the following properties:

(1) $\Theta(F)(t)$ is a closed subset of $\Re^n$, for all t;
(2) $\Theta(F)$ is measurable;
(3) $\Theta(F)(t) \subset cl\, F(t)$ for almost every t; and
(4) $\Theta(F)$ is maximal in the following sense: if U is a multi-function satisfying conditions (1)-(3), with $\Theta(F)$ replaced by U, then $U(t) \subset \Theta(F)(t)$ for almost every t.

The following theorem is due to T. Rzezuchowski (1980) and is given by Jarnik and Kurzweil (1983) in modified form:

Theorem 4.1 *Let $A \subset \Re$ be a bounded, measurable set, and $F: A \to \wp(\Re^n)$. Then, $\Theta(F)$ exists.*

Following Jarnik and Kurzweil (1983), we define an operator Φ associating pairs (F,I) with elements of $\wp(\Re^n)$. For $I = [a,b]$, F a multifunction $F: I \to \wp(\Re^n)$, and $S(F, \Delta)$ the Riemann sum for F corresponding to the partition Δ, $\Phi(F,I)$ is defined by

$$\Phi(F,I) := \{ z \in \Re^n \mid \forall\, \varepsilon > 0, \exists \text{ gauge } \delta \text{ such that } \Delta \text{ sub } \delta \Rightarrow \rho(z, S(F, \Delta)) < \varepsilon\}. \qquad \textbf{(18)}$$

Note that $\Phi(F,I) = \bigcap_{\varepsilon > 0} \bigcup_{\delta} \bigcap_{\Delta \text{ sub } \delta} W(S(F, \Delta),\varepsilon)$, where $W(S(F, \Delta),\varepsilon)$ is the open ε-neighborhood of $S(F, \Delta)$.

The following theorems and their proofs may be found in Jarnik and Kurzweil (1983):

Theorem 4.2 *Let $F: I \to \wp(\Re^n)$ be a mapping for which $F(t)$ is bounded for all $t \in I$. Then the set $\Phi(F,I)$ is closed and convex; it is compact provided F is integrably bounded.*

Theorem 4.3 *Let $F: I \to \wp(\Re^n)$ be such that $F(t)$ is bounded for all $t \in I$. Then $\Phi(F,I) = \Phi(\text{cl co } F,I)$.*

Theorem 4.4 *Let $F: I \to \wp(\Re^n)$ be measurable, integrably bounded, and let $F(t)$ be compact and nonempty for $t \in I$. Then,*

$$(\mathcal{A})\int_I F(t)\, dt = \Phi(F,I). \qquad \textbf{(19)}$$

The integral on the left side of the equality is the Aumann integral.

Jarnik and Kurzweil (1983) formulate a definition of the integral of a multifunction in the following way:

Definition 4.1 *Let $I = [a,b]$, and let $F: I \to \wp(\Re^n)$. Then*

$$\int_I F(t)\,dt := \bigcap_D \sum_{j=1}^{m} \Phi(F,[\sigma_{j-1}, \sigma_j]). \tag{20}$$

The intersection is taken over all finite decompositions D of the interval I. If $A \subset \Re$ is bounded, then for $F: A \to \wp(\Re^n)$, define $\int_A F(t)\,dt := \int_{I(A)} F_A(t)\,dt$, *where $I(A)$ is a compact interval, $A \subset I(A)$, and $F_A : I(A) \to \wp(\Re^n)$ is defined by*

$$F_A(t) = \begin{cases} F(t), & \text{for } t \in A, \\ \{0\} & \text{otherwise.} \end{cases} \tag{21}$$

A first but important observation is that, unlike the Aumann integral, this definition is direct in that it does not depend on the Kurzweil integral for point-valued mappings, and so does not depend on a selection theorem.

From this point on, when no qualification is made, the integral of a set-valued function refers to the Kurzweil integral. From the definitions, Theorems 4.2 and 4.3, it is noted that, without any assumptions on F,

$$(A)\int_I F(t)\,dt \subset \int_I F(t)\,dt, \tag{22}$$

that the integral $\int_I F(t)\,dt$ is convex and closed (compact if F is integrably bounded) and, moreover, $\int_I F(t)\,dt = \int_I cl\ co\ F(t)\,dt$. Since $F(t) \subset cl\ F(t) \subset cl\ co\ F(t)$, we also have

$$\int_I F(t)\,dt \subset \int_I cl\ F(t)\,dt \tag{23}$$

Finally, if F is measurable and integrably bounded, then

$$(\mathcal{A})\int_I cl\ F(t)\,dt = \Phi(F,I). \tag{24}$$

Consequently, if $I = [a,b]$, $a < c < b$, then by the properties of the Aumann integral, we have $\Phi(F,[a,c]) + \Phi(F,[c,b]) = \Phi(F,[a,b])$, which yields

$$\int_I F(t)\,dt = \Phi(F,I). \tag{25}$$

From the preceding results, we see that if F is a closed-valued, measurable, integrably-bounded multifunction, the Aumann and Kurzweil integrals coincide.

The main result in Jarnik and Kurzweil (1983) is the following theorem:

Theorem 4.5 *Let $A \subset \Re$ be measurable and bounded. Let $F: A \to \wp(\Re^n)$ be integrably bounded. Then,*

$$\int_A \Theta\ (cl\ co\ F)(t)\,dt \subset \int_A F(t)\,dt. \tag{26}$$

Moreover, $M := \Theta\ (cl\ co\ F)$ is measurable and integrably bounded, and we have

$$(\mathcal{A})\int_A M(t)\,dt = \int_A M(t)\,dt = \Phi\ (M_A, I(A)). \tag{27}$$

Corollary 4.6 *Let $\mathcal{K}_0^n$ denote $Conv(\Re^n) \cup \{\emptyset\}$. Suppose $F: I \to \mathcal{K}_0^n$ is integrably bounded. Then,*

$$(\mathcal{A})\int_I F(t)\,dt = \int_I F(t)\,dt. \tag{28}$$

AN EXISTENCE THEOREM FOR DIFFERENTIAL INCLUSIONS USING THE KURZWEIL INTEGRAL

The last section of Jarnik and Kurzweil (1983) is devoted to a discussion of a generalization of the classical result on the equivalence of a differential equation and the corresponding integral equation to differential inclusions.

If F: If $F: I \times \Re^n \to Conv\,\Re^n$ is a multivalued mapping, the function $x: J \to \Re^n$, $(J \subset I)$ is a solution of

$$\dot{x} \in F(t,x) \tag{29}$$

if it is absolutely continuous and

$$\dot{x}(t) \in F(t,x(t)) \text{ for a. e. } t \in J. \tag{30}$$

The set of all solutions of (5.1) is denoted by Sol F. On the other hand, *Int F* denotes the set of all functions x: $J \to \Re^n$, $J \subset I$ an interval, such that for any t, $t+h \in J$,

$$x(t+h) - x(t) \in \int_t^{t+h} F(\tau, x(\tau))\,d\tau \tag{31}$$

holds. The final theorem by Jarnik and Kurzweil (1983) is the following:

Theorem 5.1 *If $F: I \times \Re^n \to Conv\ \Re^n$ is integrably bounded, then*

$$Sol\ F = Int\ F. \quad \textbf{(32)}$$

Remark 5.1 *If one considers the conditions on $F(t,x)$ under which Davy* (1972) *proved an existence theorem for the inclusion*

$$\dot{x}(t) \in F(t,x), \quad x(t_0) = x_0, \quad \textbf{(33)}$$

then in view of Theorem 5.1, no substantial generalization is achieved from considering the integral formulation (Eqn. 31) *as a substitute for the differential inclusion. We will then consider a weakening of Davy's conditions, and proceed to consider the integral inclusion for solutions.*

For a set-valued map $F: I \times \Re^n \to \wp(\Re^n) \backslash \{\varnothing\}$, we introduce the following notation: $\mathcal{F}$ is the set-valued map with $\mathcal{F}(t,x) = cl\ co\ F(t,x)$ for $(t,x) \in I \times \Re^n$.

In this section, we assume that $I = [a,b]$, $F: I \times \Re^n \to \wp(\Re^n) \backslash \{\varnothing\}$ is a set-valued mapping satisfying the following conditions:

(1) (C1) $\mathcal{F}(t,x) \to cl\ co\ F(t,x)$ satisfies Property (Q) with respect to x, and
(2) (C2) $F(t,x)$ is integrably bounded, that is, there exists $g \in L^1(I,\Re)$ such that whenever $y \in F(t,x)$, then $|y| \le g(t)$.

Remark 5.2 *Condition* (C1) *is more general than the condition that $F(t,\bullet) : x \to F(t,x)$ satisfies Property (Q) (with respect to x). Note that whenever $F(t,\bullet) : x \to F(t,x)$ satisfies Property (Q), then $F(t,x)$ is necessarily closed and convex, and so $F(t,x) = \mathcal{F}(t,x)$. Thus, if $F(t,x)$ satisfies Property (Q) with respect to x, necessarily, condition* (C1) *holds. The converse is not true, as the following example illustrates:*

Example 5.1 Consider the multifunction *$F(x)$, such that*

$$F(x) = \begin{cases} \{-1\} & \text{if } x > 0 \\ \{-1,1\} & \text{if } x = 0 \\ \{1\} & \text{if } x < 0 \end{cases} \quad \textbf{(34)}$$

The map $F(x)$ does not satisfy Property (Q), since $F(0)$ is not convex. However, $F(0) = [-1,1]$, and it can be shown that $\mathcal{F}$ satisfies Property (Q).

With reference to condition (C2), we require integrable-boundedness because Kurzweil's result allows a reformulation of the Kurzweil integral $\int_I F(t)\,dt$ as an Aumann integral if $F(t)$ is integrably bounded. In this case,

$$\int_I F(s,x(s))\,ds = (\mathcal{A}) \int_I \Theta\ \mathcal{F}(s,x(s))\,ds. \quad \textbf{(35)}$$

We define solutions to the differential inclusion (5.5) in the Kurzweil sense in the following way:

Definition 5.1 *x is a solution to Inclusion (5.3) if and only if $x : I \to \Re^n$ is absolutely continuous and $x(t) - x(t_0) \in (\mathcal{K}) \int_{t_0}^{t} F(s,x(s))\,ds$, that is, under the assumption of integrable-boundedness,*

$$\dot{x}(t) \in \Theta(\mathcal{F}(t,x(t)) \quad a.e.\ t \in I \text{ and } x(t_0) = x_0. \quad \textbf{(36)}$$

We denote the set of solutions by $H(t_0,x_0)$.

We will make use of the following results in what follows:

Proposition 5.2 *Suppose I is a compact interval, $F: I \to \wp(\Re^n)$ is a multifunction, and $x: I \to \Re^n$ is absolutely continuous in I. Suppose almost everywhere in I. Then, almost everywhere in I.*

Proof: Define $U: I \to \wp(\Re^n)$ in the following way:

$$U(t) = \begin{cases} \{\dot{x}(t)\}, & \text{when } \dot{x}(t) < \infty \\ \varnothing, & \text{otherwise} \end{cases} \quad \textbf{(37)}$$

Observe that $U(t)$ is a closed subset of $\Re^n$ for all t, since sets which are either singletons or $\varnothing$ are closed. Moreover, U is a measurable multifunction; for if D is a closed subset of $\Re^n$, then

$$\{t \in I \mid U(t) \cap D \ne \varnothing\} = \{t \in I \mid \{\dot{x}(t) \cap D \ne \varnothing\}, \quad \textbf{(38)}$$

and the latter is measurable, because $x(.)$ is measurable. By hypothesis, since $\dot{x}(t) \in \mathcal{F}(t)$ a.e., and $F(t)$ is closed, we have

$$\{\dot{x}(t)\} = U(t) \subseteq cl\, \mathcal{F}(t) \text{ a.e.,} \quad (39)$$

(provided $\dot{x}(t) < \infty$; otherwise, $\varnothing \subseteq \mathcal{F}(t)$ since $\mathcal{F}(t)$ is itself closed.)

Thus, by the maximality of $\Theta(\mathcal{F}(t))$, we get $U(t) \subseteq \Theta(\mathcal{F}(t))$ a.e. Hence, $\dot{x}(t) \in \Theta(\mathcal{F}(t))$. *q.e.d.*

Lemma 5.3 *Suppose F has non-empty values in $P(\Re^n)$ and whenever $y \in F(t,x)$, $|y| \leq g(t)$, where g is an $\mathcal{L}^1$ function. Then, $w \in \mathcal{F}(t,x)$ also implies that $|w| \leq g(t)$.*

Proof: Let w be an arbitrary element of $\mathcal{F}(t,x)$. Then, there exists $(w_k) \subset co\, F(t,x)$ such that $w_k \to w$. For each k, $w_k \in co\, F(t,x)$. That is, $w_k = \sum_{i=1}^{nk} \lambda_i^{(k)} z_i^{(k)}$, where $z_i^{(k)} \in F(t,x)$ and $\lambda_i^{(k)} \geq 0$ for each i and $\sum_{i=1}^{nk} \lambda_i^{(k)} = 1$. Note that $|z_i^{(k)}| \leq g(t)$. Hence,

$$|w_k| = |\textstyle\sum_{i=1}^{nk} \lambda_i^{(k)} z_i^{(k)}| \leq \sum_{i=1}^{nk} |\lambda_i^{(k)} z_i^{(k)}| = \sum_{i=1}^{nk} \lambda_i^{(k)} |z_i^{(k)}| \leq (\sum_{i=1}^{nk} \lambda_i^{(k)}) g(t). \quad (40)$$

So, then, $|w_k| \leq g(t)$. As $w_k \to w$, we know that given $\varepsilon > 0$, there exists N such that $k \geq N$ implies that $|w_k - w| < \varepsilon$.

Now,

$$|w| \leq |w_k| + |w - w_k| < |w_k| + \varepsilon \text{ for large k,} \quad (41)$$

hence, $|w| < g(t) + \varepsilon$. Therefore, $|w| \leq g(t)$. *q.e.d.*

We now state and prove an extension of the Existence Theorem in Davy (1972):

Theorem 5.4. Existence of Solutions: *For a set-valued map $F: I \times \Re^n \to \wp(\Re^n)\backslash\{\varnothing\}$, suppose that F satisfies conditions ($\mathcal{C}$1) and ($\mathcal{C}$2). If we denote the set of solutions of*

$$\dot{x} \in F(t,x), \qquad x(t_0) = x_0 \quad (42)$$

in the Kurzweil sense by $H(t_0, x_0)$, then $H(t_0, x_0)$ is nonempty.

Proof: The proof follows the scheme employed by Davy (1972). Assume that $t_0 \in I$, with $I = [a,b]$. Subdivide the interval $[t_0,b]$ into k equal parts, by $t_i = t_0 + \left[i \left(\frac{b - t_0}{k} \right) \right]$. We define $x_k: [t_0,b] \to \Re^n$ inductively. First, $x_k(t_0) = x_0$. Suppose x_k is defined up to t_i, where $0 \leq i < k$. Select a measurable function $f_i: [t_i, t_{i+1}] \to \Re^n$ such that $f_i(t) \in \Theta(\mathcal{F}(t, x_k(t_i)))$ for all $t \in [t_i, t_{i+1}]$. Such a selection is possible due to the Kuratowski - Ryll-Nardzewski Theorem. Define

$$x_k(t) := x_k(t_i) + (\mathcal{L}) \int_{t_i}^{t} f_i(\tau)\, d\tau, \ \forall\, t \in [t_i, t_{i+1}]. \quad (43)$$

Then define $f: [t_0,b] \to \Re^n$ by

$$f(t) = f_i(t) \text{ for } t \in [t_i, t_{i+1}]. \quad (44)$$

Hence, $x_k(t) = x_0 + (\mathcal{L}) \int_{t_0}^{t} f$. Now, if $F(t,x)$ is integrably bounded, so too is $\mathcal{F}(t,x)$, see Lemma 5.3. Furthermore, with $\Theta(\mathcal{F}(t,x)) \subset \mathcal{F}(t,x)$, we conclude that $\Theta(\mathcal{F}(t,x))$ is also integrably bounded. This implies that $|x_k(t)| \leq |x_0| + \int_{t_0}^{b} g$. So, x_k is well-defined, and the sequence $\{x_k\}$ is bounded.

Now, $\dot{x}_k(t) = f(t)$ a.e. $t \in [t_0,b]$. Therefore, $|\dot{x}_k(t)| \leq g(t)$ a.e. $t \in [t_0,b]$. Therefore, $\{x_k\}$ is an equicontinuous family.

Define $y_k: [t_0,b] \to \Re^n$ by

$$y_k(t) := x_k(t_i) \text{ if } t \in [t_i, t_{i+1}]. \quad (45)$$

Then, $\dot{x}_k(t) \in \Theta(\mathcal{F}(t, y_k(t)))$ a.e. $t \in [t_0,b]$. Consider the sequence $\{x_k\}$. It is bounded and equicontinuous. Thus, by Ascoli's Theorem, it has a convergent subsequence, and we denote this subsequence also by $\{x_k\}$. We then have $x_k \to x \in C([t_0,b])$, where the latter denotes the set of continuous functions on $[t_0,b]$. By Theorem 2.1, x is absolutely continuous and

$$\dot{x}(t) \in \bigcap_{i=1}^{\infty} cl\, co \bigcup_{k=i}^{\infty} \dot{x}_k(t) \subseteq \bigcap_{i=1}^{\infty} cl\, co \bigcup_{k=i}^{\infty} \Theta(\mathcal{F}(t, y_k(t))) \quad (46)$$

a. e. $t \in [t_0,b]$, since $\dot{x}_k(t) \in \Theta(\mathcal{F}(t, y_k(t)))$.

But $y_k(t) \to x(t)$. We next observe that $\Theta(\mathcal{F}(t,x) \subset \mathcal{F}(t,x)$, and invoke Property (Q), satisfied by $\mathcal{F}(t,x)$ with respect to x, to conclude that

$$\dot{x}(t) \in \bigcap_{i=1}^{\infty} cl\ co \bigcup_{k=i}^{\infty} \mathcal{F}(t,y_k(t)) \subseteq \mathcal{F}(t,x(t)). \quad \textbf{(47)}$$

So, $x(t) \in \mathcal{F}(t,x(t))$ a.e. $t \in [t_0,b]$. We then apply Proposition 5.2 to conclude that $x(t) \in \Theta(\mathcal{F}(t,x(t)))$ almost everywhere in $[t_0,b]$. Furthermore, as $x_k(t_0) = x_0$ for all k and $x_k \rightarrow x$, we get $x(t_0) = x_0$.

Similarly, we can find $x: [a,t_0] \rightarrow \Re^n$ such that $x(t_0) = x_0$ and $x(t) \in \Theta(\mathcal{F}(t,x(t)))$ a.e. $t \in [a,t_0]$. Concatenating these two functions, we have $x:[a,b] \rightarrow \Re^n$ and $x \in H(t_0, x_0)$. Thus, $\int +(t_0, x_0) \neq \varnothing$ q.e.d.

Example 5.2. *Consider the multifunction*

$$F(x) = \begin{cases} \{-1\} & \text{for } x > 0 \\ \{-1, 1\} & \text{for } x = 0 \\ \{1\} & \text{for } x < 0 \end{cases} \quad \textbf{(48)}$$

Take the initial problem $\dot{x} \in F(x)$, $x(0p) = 0$. *Observe that F is upper semicontinuous, and that the solutions cannot leave the initial value. But now,* $x = 0$ *is not a solution, hence there is no solution to the initial value problem in the usual sense. However,* $\mathcal{F}(x)$ *satisfies Property (Q), and moreover, F is integrably bounded. Hence, there is a solution in the Kurzweil sense; one such solution is* $x = 0$.

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