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Transitive Perfect Colorings of 2-Uniform Tilings

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ABSTRACT

In this work, a method to determine the nontrivial colorings of perfect and transitive 2-uniform tilings is presented. This method has been applied to determine all nontrivial transitive perfect colorings of 2-uniform tilings that use the least number of colors. In addition, the equivalence of the colorings obtained was also ascertained.

Keywords: Perfect colorings, 2-uniform tilings, equivalent colorings

INTRODUCTION

Numerous tilings of the plane by regular polygons have long been known, such as the regular tilings 3^6, 4^4, and 6^3 as well as the 8 semi-regular tilings 3.12^2, 4.6.12, 4.8^2, 3.4.6.4, 3.6.3.6, $3^4.6$, $3^2.4.3.4$, and $3^3.4^2$, as illustrated by Grünbaum and Shepard (1987). These tilings are also known as Archimedean tilings. Given any pair of vertices of the tiling, the Archimedean tilings would exhibit symmetry (translation, rotation, reflection, or glide reflection) that sends one vertex to the other. That is, the vertices of an Archimedean tiling form one transitivity class under the action of the symmetry group of the tiling. For this reason, the arrangement of polygons about a vertex is the same for every vertex of an Archimedean tiling. For example, 3.4.6.4 means that a vertex is surrounded in cyclic order by a triangle (3-gon), a square (4-gon), a hexagon (6-gon) and a square. On the other hand, 6^3 is just 6.6.6, meaning a vertex is surrounded by three hexagons.

Lesser known are the 2-uniform tilings, which are edge to edge tilings by regular polygons and where vertices of the tiling form two transitivity classes. These

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tilings have 20 types, as shown in Figure 1. The enumeration of these tilings is attributed to Krötenheerdt (1969). Each of the 2-uniform tilings is described through the vertex types of the two transitivity classes. For example, (3^2.4.3.4; 3.4.6.4) describes a 2-uniform tiling where the vertices are of types 3^2.4.3.4, and 3.4.6.4.

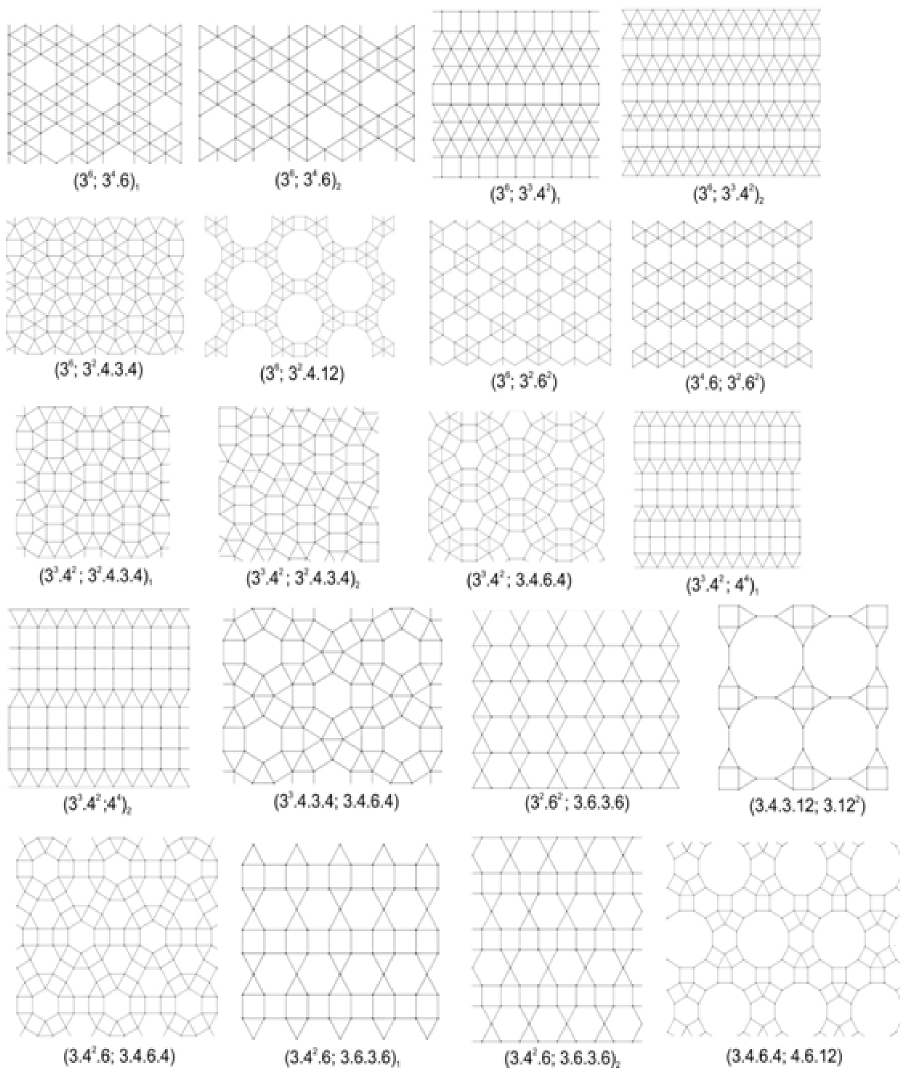


Figure 1. The twenty 2-uniform tilings in the Euclidean plane.

In this study, the colorings of 2-uniform tilings, which fall under the theory of color symmetry, were considered. The basic problem in color symmetry is the classification of symmetrically colored symmetrical patterns. The work of Schwarzenberger (1984) provides a compendium of results on color symmetry, spanning decades of works. Senechal (1988) discussed results of interest and posed some problems on color symmetry.

The paper of Rapanut (1988) provided useful results on subgroups of the seventeen plane crystallographic groups. In his paper, Roth (1993) determined that the minimum number n of colors that suffice to color any multipattern with an associated symmetry group is $2 \leq n \leq 25$.

More recent works on coloring symmetrical patterns in the case of hyperbolic plane patterns have been done by De Las Peñas, Felix, and Laigo (2006). Frettlöh (2008) listed possible values for perfect k-colorings of some hyperbolic regular and Laves tilings. Felix and Loquias (2008) worked on semiperfect colorings. Precise perfect colorings were studied by Santos and Felix (2011). A study on transitive perfect colorings on semi-regular tilings was done by Gentuya (2013).

PRELIMINARIES

Let X be a set of objects in the plane and G the symmetry group of X. A *coloring* of X (using n colors $c_1, c_2, \ldots, c_n$) is a surjective or onto function from X to $\{c_1, c_2, \ldots, c_n\}$. The coloring results in a partition $P = \{P_1, P_2, \ldots, P_n\}$ of X where two elements x and y in X are assigned the same color c_i if and only if they are elements of the same set P_i. We may therefore treat a coloring as a partition P of X. If $gP = P$ for every $g \in G$, we say that the partition P is G-invariant and that the associated coloring is *perfect*. We also say that each $g \in G$ induces a permutation of the colors $\{c_1, c_2, \ldots, c_n\}$.

A special class of perfect colorings of X is the class of transitive perfect colorings of X. A perfect coloring of X is *transitive* if G acts transitively on the set of colors $\{c_1, c_2, \ldots, c_n\}$, i.e., if c_i and c_j are any two colors in $\{c_1, c_2, \ldots, c_n\}$ there is an element in G that sends c_i to c_j. Thus, not only is each symmetry in G associated with a unique permutation of the colors of the colored set, but given any two colors, the pattern formed by elements of one color is congruent to the pattern formed by elements of the other color. Hence, the colored pattern may be thought of as a disjoint union of colored subpatterns that are congruent to each other.

Consider two colorings of the same set X and the corresponding colored patterns arising from the two colorings. The two colorings are said to be *equivalent* if one of the colored patterns may be obtained from the other colored pattern by (1) a bijection from the set of colors used in the first coloring to the set of colors used in the second coloring, (2) a symmetry in the symmetry group G of X, or (3) a combination of (1) and (2). This definition of equivalence is adapted from Roth (1982).

The concepts are illustrated using the colored patterns in Figure 2. In Figure 2(a), X = {1, 2, 3, 4, 5, 6, 7, 8} is a set of eight points with symmetry group $G = \langle a, b \rangle = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\} \cong D_4$ where a is a 90°-counterclockwise rotation about the center of the configuration and b is a mirror reflection about the horizontal line passing through the center of a. Figure 2(b) exhibits a transitive perfect coloring of X. The coloring corresponds to the partition $P = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$, where the points in {1, 2}, {3, 4}, {5, 6}, and {7, 8} are colored red (*R*), blue (*B*), green (*G*), and yellow (Y), respectively. The 90° rotation a results in the permutation (*RYGB*), whereas the reflection b results in the permutation (*BY*). Given any two colors in the set {*R, Y, G, B*}, there is a symmetry in G that sends one color to the other color. The coloring of X in Figure 2(c) is not perfect. The only elements of G that induce a permutation of the colors are e, a^2, b, and a^2b. Figure 2(d) and

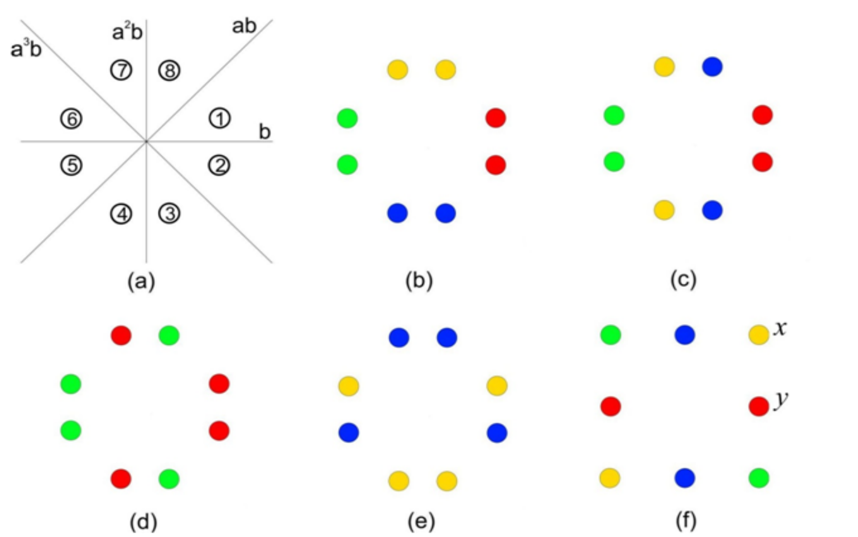


Figure 2. The set X consisting of eight distinct points with symmetry group $G = \langle a, b \rangle \cong D_4$ showing (a) the mirror elements of G, (b) a transitive and perfect coloring, (c) a non-transitive and non-perfect coloring, (d)–(e) equivalent colorings, and (f) the set X' of eight points with symmetry group $G = \langle a, b \rangle \cong D_4$ with a perfect but non-transitive coloring.

Figure 2(e) exhibit equivalent colorings of X. Using the bijection *red*→blue and green → yellow and then applying the 90° rotation a we obtain the colored pattern in Figure 2(e) from the colored pattern in Figure 2(d).

In Figure 2(f), we illustrate a coloring of a set X' which is perfect but not transitive. The symmetry group of X' is also $G = \langle a, b \rangle \cong D_4$. Let x denote the upper right hand corner point and the point immediately below as shown in the figure. The G-orbit of x refers to the set $Gx = \{gx : g \in G\}$ and consists of the images of x under the elements of G. The set of corner points of X' is the G-orbit of x whereas the G-orbit of y is the remaining set of points in X'. The rotation a induces the permutations $(RB)\,(GY)$ and the reflection b induces the permutation (GY). Hence, the coloring is perfect. The coloring is not transitive because there is no symmetry in G that will send the color *yellow* to *blue*.

COLORING FRAMEWORK

In this study, nontrivial transitive perfect colorings of 2-uniform tilings were considered. The approach of Felix (2011) where a coloring of a set is treated as a partition of the set was used. We made use of the theorem described below. In the theorem, $Stab_G(x)$ denotes the set $\{g \in G : gx = x\}$ and is called the *stabilizer* in G of x.

Theorem. *Let X be a set and let G be a group acting transitively on X .*

1. *If $\{X_i\}_{i=1}^m$ is a coloring of X for which G permutes the colors, then for every $x \in X$, there exists $J \leq G$ such that $Stab_G(x) \leq J$ and the coloring is described by the partition*

$$\{\, gJx : g \in G \,\}.$$

2. *Let $x \in X$ and $J \leq G$ such that $Stab_G(x) \leq J$ and $[G : J] = n < \infty$. Then*

$$P = \{gJx : g \in G\}$$

is a coloring of X with n colors for which G permutes the colors.

(See Evidente, 2012 for the proof).

Remark: The above theorem determines all perfect colorings of X on the assumption that G acts transitively on the set X. The partition P above corresponds to a coloring of X that is perfect and transitive.

Based on the theorem, a procedure for arriving at nontrivial transitive perfect colorings of a 2-uniform tiling where the number of colors used is minimal is described.

METHOD FOR DETERMINING TRANSITIVE PERFECT COLORINGS OF A 2-UNIFORM TILING USING THE LEAST NUMBER OF COLORS

1. Given a 2-uniform tiling, let G denote the symmetry group of the tiling. This group G is a plane crystallographic group.

2. The set X of tiles of the tiling is partitioned into a finite number of G-orbits X_i, $i = 1, 2, \ldots, m$.

3. For each G-orbit, X_i, $i = 1, 2, \ldots, m$, obtain a G-orbit representative $x_i \in X_i$.

4. Obtain $Stab_G(x_i)$, $i = 1, 2, \ldots, m$. This group is a finite group, which is cyclic $(\cong C_n)$ or dihedral $(\cong D_n)$.

5. Look for a proper subgroup J of G of least index such that for each $i = 1, 2, \ldots, m$, a conjugate of $Stab_G(x_i)$ is contained in J.

Assume there is a subgroup J that was obtained in 5.

6. Obtain the J-orbits of tiles of the tiling.

7. For each $i = 1, 2, \ldots, m$, choose a tile $t_i \in X_i$ such that $Stab_G(t_i) \leq J$ or equivalently $Stab_J(t_i) = Stab_G(t_i)$.

8. Form the set $T = \{t_1, t_2, \ldots, t_m\}$ and denote by JT the set $Jt_1 \cup Jt_2 \cup \ldots \cup Jt_m$; i.e., JT is the union of the J-orbits Jt_i, $i = 1, 2, \ldots, m$.

9. Let $\{g_1, g_2, \ldots, g_n\}$ be a complete set of left coset representatives of J in G.

10. The partition $\{g_i JT \mid i = 1, 2, \ldots, n\}$ describes a nontrivial transitive perfect coloring of the tiling using n colors $c_1, c_2, \ldots, c_n$. The coloring is given by the assignment $g_i JT \longrightarrow c_i$, $i = 1, 2, \ldots, n$; i.e., the tiles in $g_i JT$ are all colored (assigned the color) c_i.

For the basic ideas involved in formulating the procedure, some explanations are provided below.

First, G-orbits X_i was considered to be independently colored. Based on the theorem, the coloring or partition $P = \{gJ_i x_i : g \in G\}$ where $x_i \in X_i$ and $Stab_G(x_i) \leq J_i \leq G$ was used. Note that if $x_i, y_i \in X_i$, there exists $g \in G$ such that $y_i = gx_i$ and thus $Stab_G(y_i) = gStab_G(x_i)g^{-1}$; i.e., $Stab_G(x_i)$ and $Stab_G(y_i)$ are conjugate. The number of colors corresponding to the partition $P = \{gJ_i x_i : g \in G\}$ is given by $[G : J_i]$. Since using the least number of colors in the coloring is preferred, the same set $C = \{c_1, c_2, \ldots, c_n\}$ of colors was used in coloring the G-orbits $X_i, i = 1, 2, \ldots, m$, and thus we take $J_i = J$ for $i = 1, 2, \ldots, m$ for some J where $Stab_G(x_i) \leq J \leq G, i = 1, 2, \ldots, m$. If instead of $x_i \in X_i$, $y_i \in X_i$ was used to represent the G-orbit X_i then $Stab_G(y_i) \leq J \leq G$ should be obtained. However, $Stab_G(y_i) = gStab_G(x_i)g^{-1}$ if $y_i = gx_i$ for some $g \in G$. $Stab_G(y_i) \leq J \leq G$ is required.

All of the transitive perfect colorings of 2-uniform tilings were looked into, with the least number of colors (which were finite) and in all cases, a J subgroup was found. Otherwise, if no proper subgroup J of G was found, then the option will be that $J = G$, and the coloring will be trivial.

The procedure used to arrive at the results for four of the twenty 2-uniform tilings, namely, the tilings $(3^2.4.3.4; 3.4.6.4)$, $(3^3.4^2; 3^2.4.3.4)_1$, $(3^3.4^2; 3^2.4.3.4)_2$, and $(3^6; 3^2.4.3.4)$ is illustrated as follows.

The 2-Uniform Tiling $(3^2.4.3.4; 3.4.6.4)$

Consider the 2-uniform tiling $(3^2.4.3.4; 3.4.6.4)$ given in Figure 1. If we let G denote the symmetry group of the tiling, then $G = \langle u, v, a, r \rangle \cong p6m$, where u, v are two linearly independent translations, a is a six-fold rotation centered at a hexagonal tile, and r is a reflection with symmetry axis passing through the center of a. These are shown in Figure 3 together with G-orbits of tiles where tiles of the tiling belonging to the same G-orbit have the same color.

There are four G-orbits of tiles of the tiling: X_1, the set of hexagons; X_2, the set of squares; X_3, the set of triangles whose sides are sides of squares, and; X_4, the set of triangles that share one side with another triangle.

Without loss of generality, tiles $1 \in X_1$, $2 \in X_2$, $3 \in X_3$, and $4 \in X_4$ can be chosen, as given in Figure 3. The stabilizer of the hexagonal tile 1 in X_1, generated by the six-fold rotation a and the reflection r, is isomorphic to D_6. For the square tile 2 in X_2, the stabilizer is isomorphic to D_1 generated by the reflection r. The triangular tile 3 in X_3 has a stabilizer generated by a three-fold rotation about its center and the reflection r. The subgroup is isomorphic to D_3. The stabilizer of the triangular tile 4 is generated by a reflection r and is isomorphic to D_1.

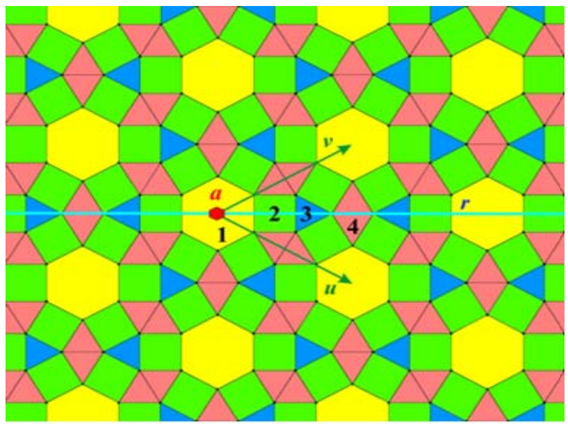


Figure 3. The (3^2.4.3.4; 3.4.6.4) tiling showing the generators and the distinct G-orbits with the tiles $1 \in X_1$, $2 \in X_2$, $3 \in X_3$, and $4 \in X_4$.

To find the subgroup $J \leq G$ of smallest index in G that yields a nontrivial transitive perfect coloring of the (3^2.4.3.4; 3.4.6.4) tiling, a plane crystallographic group H that contains subgroups of type D_6, D_3, and D_1 must be identified. Among the subgroups of G isomorphic to H is the subgroup J, which is needed in this instance. The results of Rapanut (1988) indicate that $H \cong p6m$. Moreover, the subgroups of G of type $p6m$ are of index n^2 or $3n^2$, where n is a natural number. This gives the possible indices 1, 3, 4, 9, and so on. Since the least possible index n is being determined such that the coloring is nontrivial, the subgroup $J_0 = \langle u^2v^{-1}, u^{-1}v^2, a, r \rangle \leq G$ of index 3 is first considered. Figure 4 shows a unit cell corresponding to the subgroup J_0. For simplicity, a unit cell of the tilings will be looked into. Schattschneider (1978) can be referred to for the unit cells corresponding to the 17 plane crystallographic groups and the symbols used to denote centers of rotations.

An inspection of the unit cell shows that no 3-fold rotation of J_0 stabilizes a triangle in X_3 hence another low index subgroup must be considered. If we let $J = \langle u^2, v^2, a, r \rangle$, a subgroup of index 4 in G, each G-orbit splits into 2 or more J-orbits. As shown in Figure 5, there are two J-orbits of hexagons, three J-orbits of squares, two J-orbits of triangles in and three J-orbits of triangles in X_4.

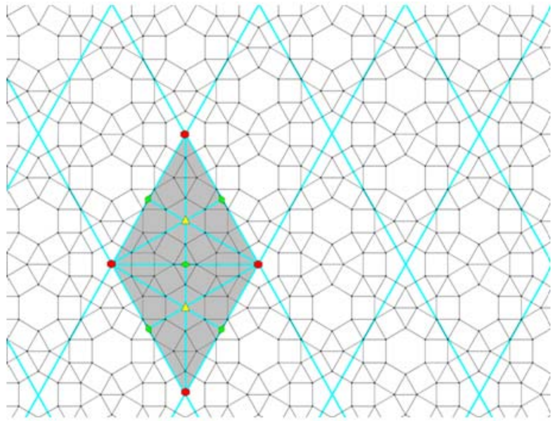


Figure 4. The $(3^2.4.3.4;\ 3.4.6.4)$ tiling with the stabilizers in J_0 shown in a unit cell (shaded region).

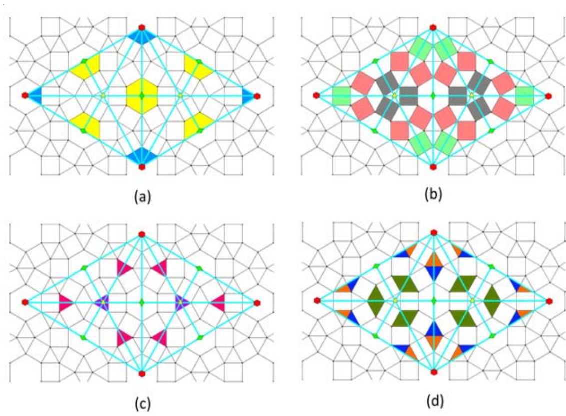
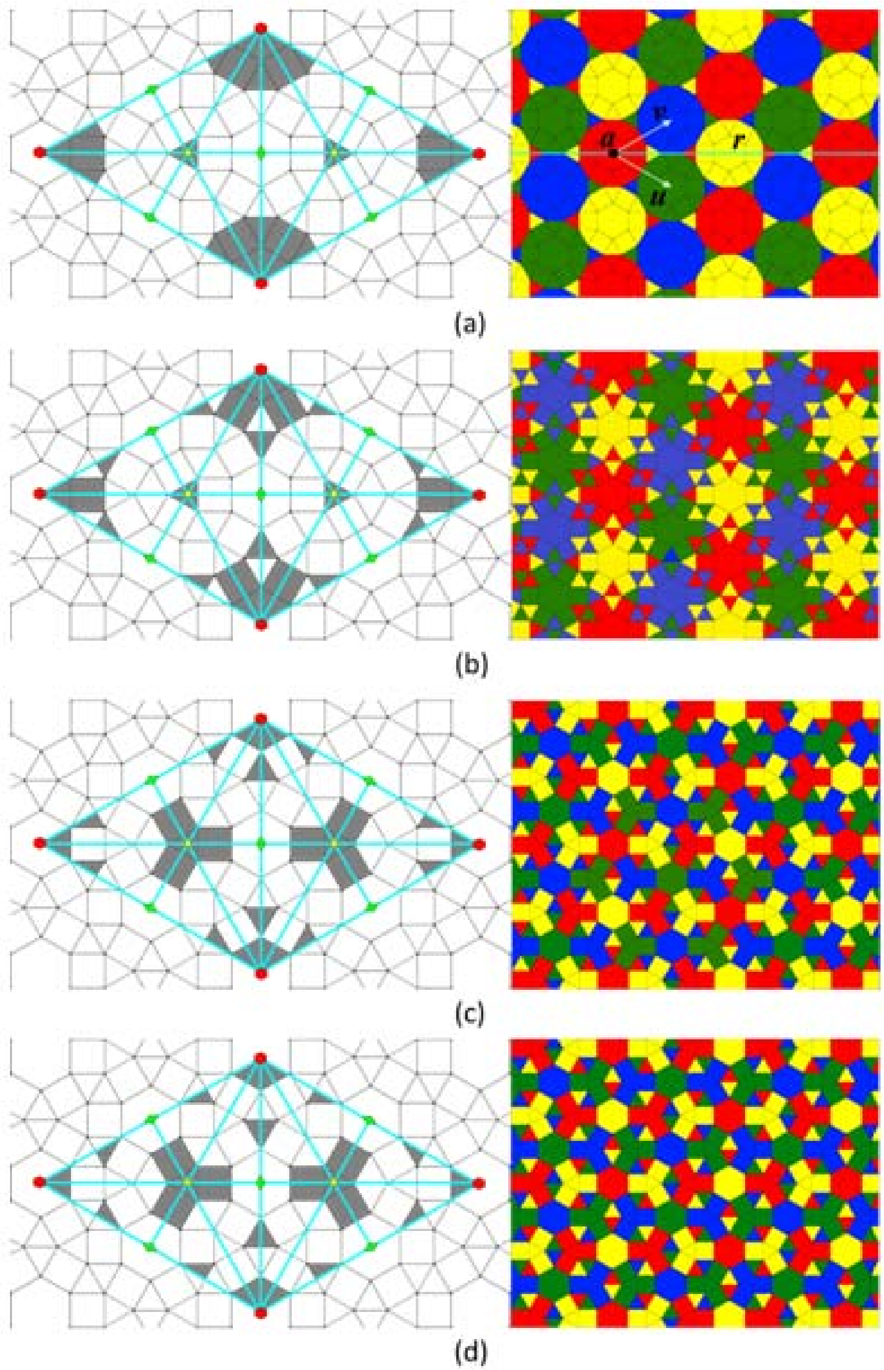


Figure 5. The J-orbits of the tiles of the tiling in (a) X_1, (b) X_2, (c) X_3, and (d) X_4.

Next, the set $T = \{t_1, t_2, t_3, t_4\}$ is formed, where $t_i \in X_i$ and $Stab_G(t_i) \leq J$. Any blue hexagonal tile in X_1 can be chosen as t_1, any grey or green square tile in X_2 can be chosen as t_2 , any purple triangular tile in X_3 can be chosen as t_3, and any orange or blue triangular tile in X_4 can be chosen as t_4. Note that in the J-orbits of tiles of the tiling in each X_i , $Stab_G(t_i) \leq J$. Form the partition, $P = \{JT, uJT, vJT, uvJT\}$. Assigning distinct colors to each g_iJT results in a transitive perfect coloring of the tiling using only four colors. Considering all possible combinations of tiles for T results in exactly four inequivalent transitive perfect 4-colorings of the $(3^2.4.3.4;\ 3.4.6.4)$ tiling, as given in Figure 6(a)–(d).



v,

Figure 6. The four possible partitions for the (3^2.4.3.4; 3.4.6.4) tiling and their corresponding inequivalent transitive perfect 4-colorings.

In each of the 4-colorings generated, color permutations are given in Table 1 (where the colors red, green, blue, and yellow are denoted by R, G, B, and Y, respectively).

Table 1. The color permutations corresponding to generators of *G*

Generator	Color Permutation
u	*(RG) (BY)*
v	*(RB (GY)*
a	*(BYG)*
r	*(BG)*

The 2-Uniform Tilings and $(3^3.4^2.3^2.4.3.4)_1$ and $(3^3.4^2.3^2.4.3.4)_2$

Consider the 2-uniform tiling $(3^3.4^2.3^2.4.3.4)_1$ with symmetry group $G = \langle u, v, a, r \rangle \cong p4g$, where u and v are two linearly independent translations, a is a 4-fold rotation and r is a reflection with symmetry axis not passing the center of rotation a, as shown in Figure 7.

There are four G-orbits of tiles of the tiling: (1) the squares that share no side with other squares, (2) the squares that share one side with another square, (3) the triangles that share exactly one side with a square, and (4) the triangles that share two sides with squares. The stabilizers in G for a tile in each G-orbit are isomorphic to C_4, D_1, D_1, and C_1, respectively. The proper subgroup J of G that contains them must be isomorphic to $p4g$. The subgroup J of least possible index n^2 where n is a natural number is $\langle u^3, v^3, r, ua \rangle$, as generated by GAP and $[\, G : J \,] = 9$

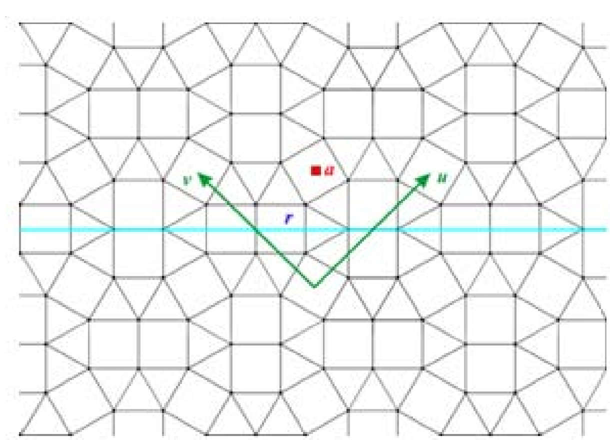


Figure 7. The $(3^3.4^2;3^2.4.3.4)_1$ tiling with the generators u, v, a, and r.

Form $T = \{t_1, t_2, t_3, t_4\}$ where $Stab_G(t_i) \leq J$ for $i = 1, 2, 3, 4$. Observe that in Figure 8(a) any yellow square tile can be chosen as t_1, in (b) t_2 can be chosen from any of the square tiles of colors red, pink, or peach, in (c) t_3 can be chosen from any of the triangular tiles of the tiling of colors orange, purple, or green, and in (d) t_4 can be any of the nine colored triangular tiles of the tiling. From all the possible choices of tiles for $\{t_1, t_2, t_3, t_4\}$, $1 \cdot 3 \cdot 3 \cdot 9 = 81$ inequivalent nontrivial transitive perfect colorings of the $(3^3.4^2.3^2.4.3.4)_1$ tiling is obtained. It may be checked that these colorings are inequivalent.

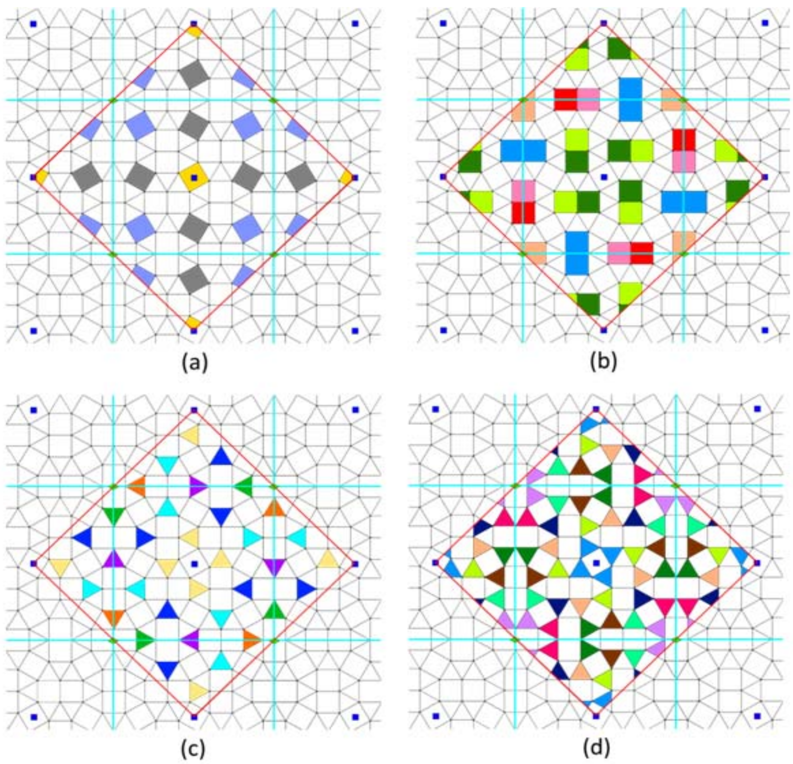


Figure 8. The J-orbits of square and triangular tiles t_i with $Stab_G(t_i) \leq J$.

One such transitive perfect coloring of the $(3^3.4^2.3^2.4.3.4)_1$ tiling using nine distinct colors is shown in Figure 9.

The 2-uniform tiling $(3^3.4^2.3^2.4.3.4)_2$ has the same vertex types $3^3.4^2$ and $3^2.4.3.4$ but its symmetry group $G = \langle u, v, p, q \rangle \cong pgg$, where u and v are two linearly independent translations, p is a glide reflection, and q is a glide reflection with glide axis perpendicular to the glide axis of p, as shown in Figure 10.

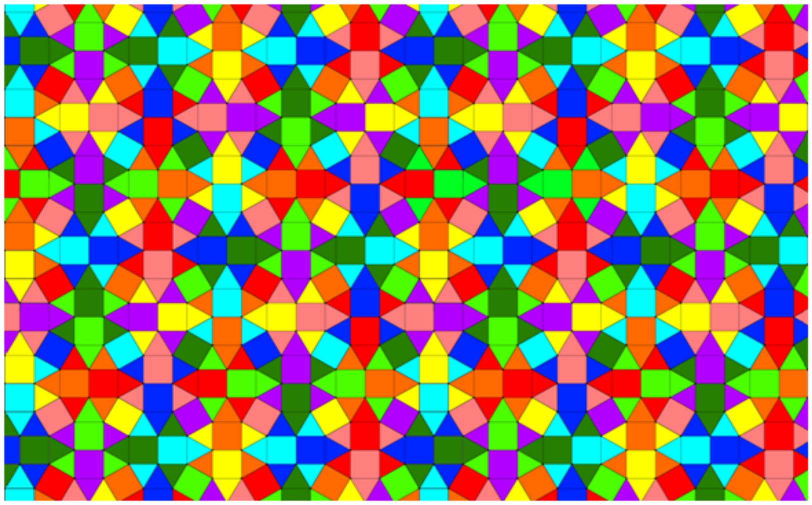


Figure 9. A transitive perfect 9-coloring of the $(3^3.4^2;\, 3^2.4.3.4)_1$ tiling.

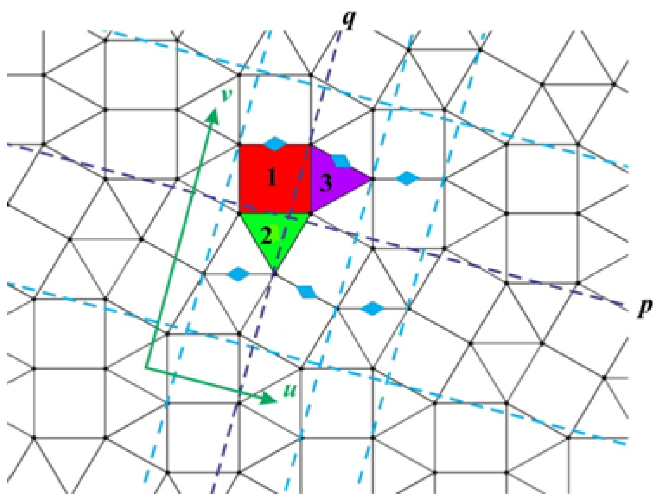


Figure 10. The $(3^3.4^2;\, 3^2.4.3.4)_2$ tiling with its generators and tiles belonging to the three G-orbits.

There are three G-orbits of tiles of the tiling: X_1 consisting of squares, X_2 consisting of triangles that share two sides with squares, X_3 and consisting of the triangles not included in X_2. The stabilizers of the tiles of the tiling in each G-orbit are all isomorphic to C_1. This is contained in any subgroup of G. Thus, we only need subgroups of least possible index greater than 1. Using GAP, three subgroups of index 2 are obtained, and these are given in Table 2.

Table 2. Subgroups of $G = \langle u, v, p, q\rangle$ of index 2

Subgroup	Index	Symmetry Group
$\langle p, q^{-2}\rangle$	2	pg
$\langle p^{-2}, q\rangle$	2	pg
$\langle p^{-2}, p^{-1}q, p^{-1}q^{-1}\rangle$	2	$p2$

If $J = \langle p, q^{-2}\rangle = \langle p, q^{2}\rangle$, each G-orbit splits into two J-orbits, as shown in Figure 11. Form $T = \{t_1, t_2, t_3\}$, where t_i is in X_i for each i = 1, 2, 3 and $Stab_G(t_i) \leq J$. In Figure 11(a), t_1 can be any yellow square tile or grey square tile. In (b), t_2 can be any orange tile or purple tile, and in (c), t_3 can be any pink tile or blue tile. These give eight possible combinations for the set T.

The subgroup $J \leq G$ is of index 2 in G and we have $G = J \cup hJ$, where h is the 2-fold rotation in G whose center is shown in Figure 11. If a set of fixed tiles for $T = \{t_1, t_2, t_3\}$ and the half-turn h in G are considered, the partition $\{JT, hJT\}$ could be obtained. Assigning the color red to JT and the color green to hJT results in a transitive perfect 2-coloring of the $(3^3.4^2; 3^2.4.3.4)_2$ tiling. Nevertheless, it should be noted that

$$h\{JT, hJT\} = \{hJT, h^2JT\} = \{hJT, JT\}.$$

That is, the symmetry $h \in G$ maps the partition $\{JT, hJT\}$ to the partition $\{hJT, JT\}$. Hence, the coloring described by $\{JT, hJT\}$ is equivalent to the coloring described by $\{hJT, JT\}$.

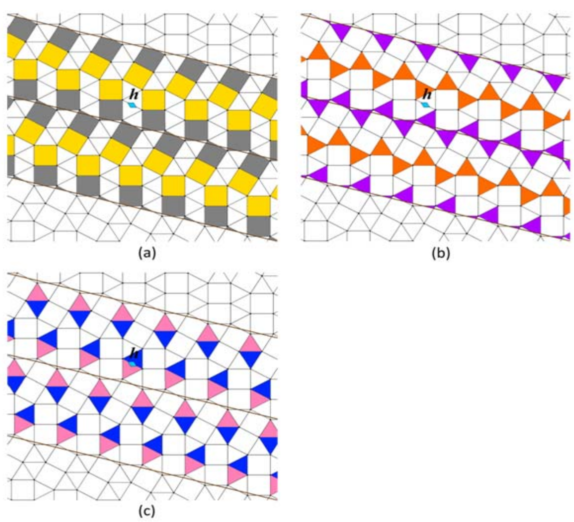


Figure 11. J-orbits of tiles of the tiling in (a) X_1, (b) X_2, (c) X_3 where $J = \langle p, q^2\rangle$ and a half-turn h in G.

In turn, this reduces the possible number of nontrivial transitive perfect colorings to four instead of eight, as presented in Figures 12(a)-(h). The coloring in (a) is equivalent to (b), (c) is equivalent to (d), (e) is equivalent to (f), and (g) is equivalent to (h).

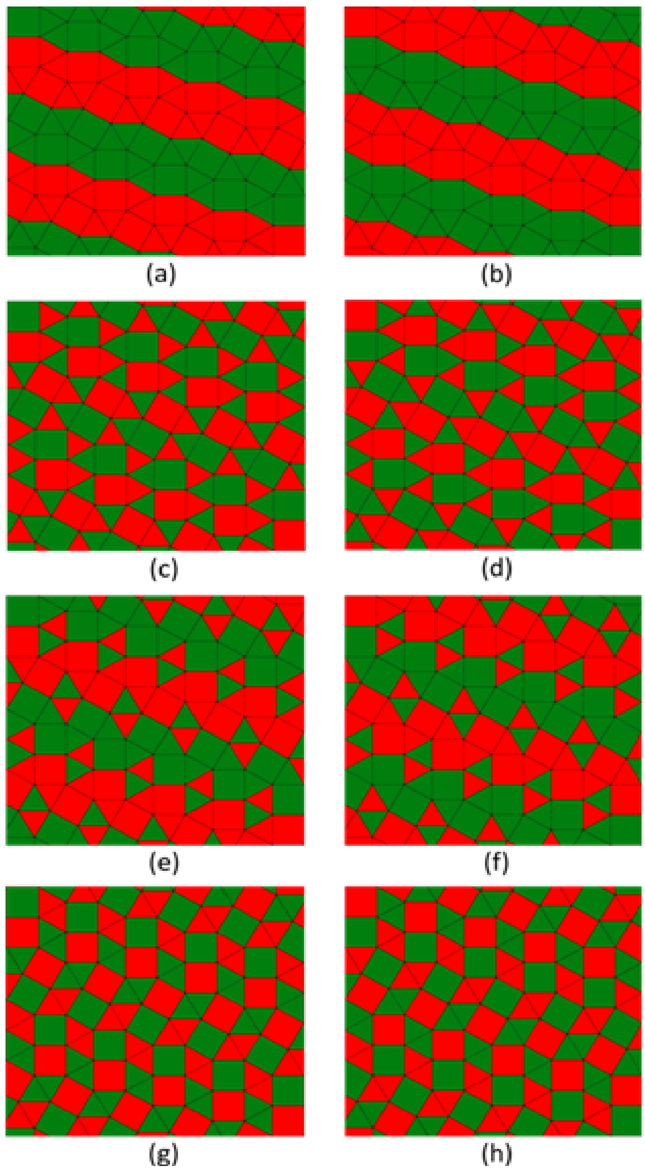


Figure 12. The eight transitive perfect 2-colorings of $(3^3.4^2;\, 3^2.4.3.4)_2$ when $J = \langle p, q^2 \rangle$.

Similarly, when we let $J = \langle p^{-2}, q\rangle = \langle p^2, q\rangle$, each G-orbit of tiles of the tiling splits into 2 J-orbits and results into four inequivalent transitive perfect 2-colorings, as seen in Figure 13.

If we let $J = \langle p^{-2}, p^{-1}q, p^{-1}q^{-1}\rangle = \langle p^2, p^{-1}q,\ p^{-1}q^{-1}\rangle$, each G-orbit of tiles of the tiling also splits into two J-orbits. The resulting 2-colorings are shown in Figure 14. In all, there are 12 inequivalent transitive perfect 2-colorings of the $(3^3.4^2;\ 3^2.4.3.4)_2$ tiling.

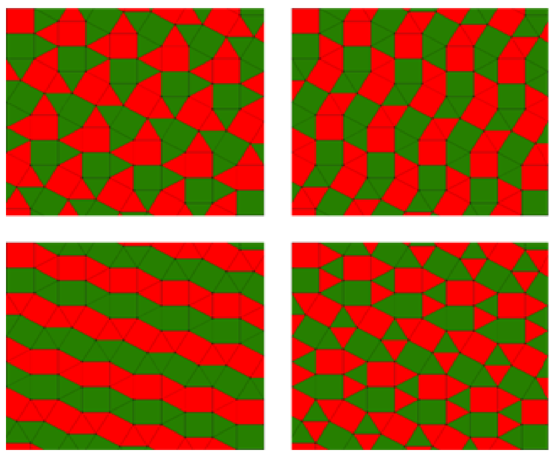


Figure 13. The four inequivalent transitive perfect 2-colorings of $(3^3.4^2;\ 3^2.4.3.4)_2$ when $J = \langle p^2, q\rangle$.

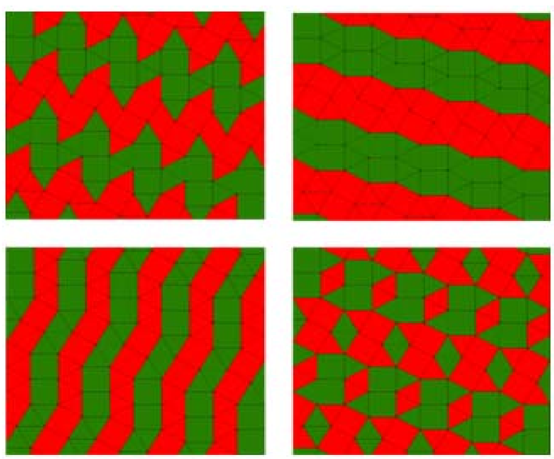


Figure 14. The four inequivalent transitive perfect 2-colorings of $(3^3.4^2;\ 3^2.4.3.4)_2$ when $J = \langle p^2, p^{-1}q,\ p^{-1}q^{-1}\rangle$.

The 2-Uniform Tiling (3^6; $3^2.4.3.4$)

Using the method discussed, five inequivalent transitive perfect 25-colorings of (3^6; $3^2.4.3.4$) were obtained. The least number of colors that can be used to color the tiling is 25. One such coloring is shown in Figure 15 where tiles of the same number are assigned the same color. Figure 16 indicates the remaining four transitive perfect 25-colorings of (3^6; $3^2.4.3.4$).

Applying the method for finding nontrivial transitive perfect colorings to all 2-uniform tilings, results were obtained, as summarized in Table 3. The patterns for all of the inequivalent colorings in each of the twenty 2-uniform tilings were illustrated.

The results show that if n is the least number of colors needed in coloring a 2-uniform tiling in such a way that it is nontrivial, transitive, and perfect then $2 \leq n \leq 25$. This result is expected based on the work of Roth (1993).

Applying the method to 3-uniform tilings, i.e., tilings by regular polygons where the vertices of the tiling form three transitivity classes, is also of interest. Additional insights may be acquired from looking at transitive perfect colorings of 3-uniform tilings. The complete list of drawings for the 61 3-uniform tilings are found in Chavey (1989).

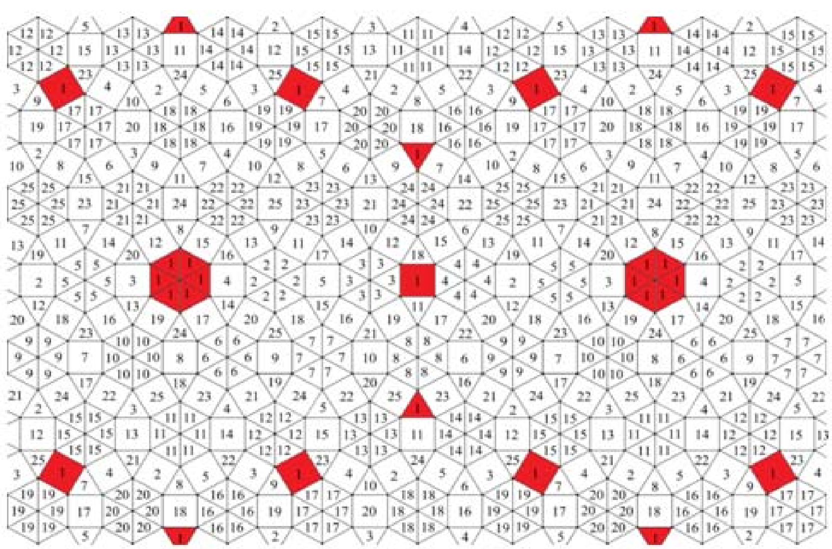


Figure 15. A transitive perfect 25-coloring of the (3^6; $3^2.4.3.4$) tiling.

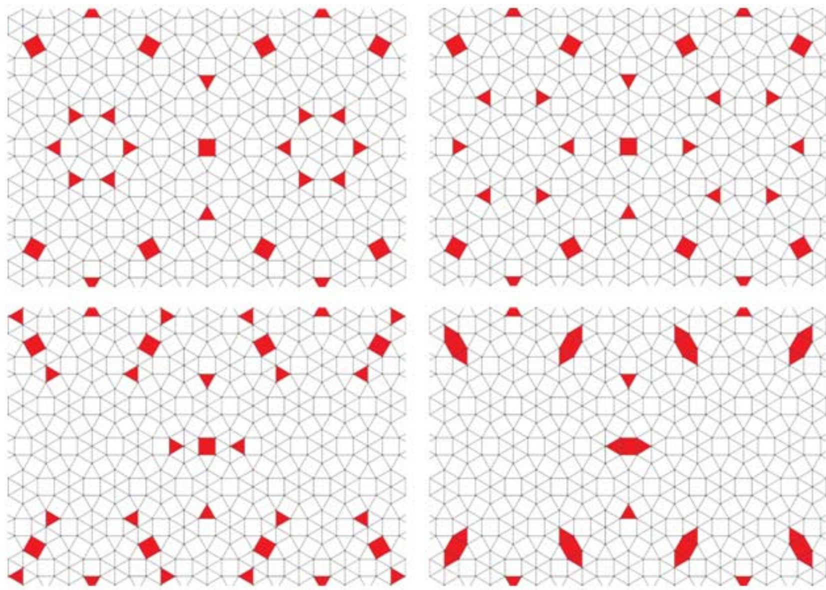


Figure 16. The other four motifs of a 25-coloring of the (3^6; 3^2.4.3.4) tiling.

Table 3. The 2-uniform tilings with their corresponding symmetry groups and the least number of colors needed for generating transitive perfect colorings

2-Uniform Tilings	Symmetry Group	Least Number n of Colors	Number of Inequivalent n-Colorings
$(3^6; 3^4.6)_1$	p6	4	64
$(3^6; 3^4.6)_2$	p6m	3	1
$(3^6; 3^3.4^2)_1$	cmm	2	8
$(3^6; 3^3.4^2)_2$	pmm	2	4
$(3^6; 3^2.4.3.4)$	p6m	25	5
$(3^6; 3^2.4.12)$	p6m	3	1
$(3^6; 3^2.6^2)$	p6m	2	2
$(3^4.6; 3^2.6^2)$	cmm	2	2
$(3^3.4^2; 3^2.4.3.4)_1$	p4g	9	81
$(3^3.4^2; 3^2.4.3.4)_2$	pgg	2	12
$(3^3.4^2; 3.4.6.4)$	p6m	4	4
$(3^3.4^2; 4^4)_1$	cmm	2	2
$(3^3.4^2; 4^4)_2$	cmm	2	4
$(3^3.4.3.4; 3.4.6.4)$	p6m	4	4
$(3^2.6^2; 3.6.3.6)$	pmm	2	2
$(3.4.3.12; 3.12^2)$	p4m	9	3
$(3.4^2.6; 3.4.6.4)$	p6m	25	25
$(3.4^2.6; 3.6.3.6)_1$	pmm	3	3
$(3.4^2.6; 3.6.3.6)_2$	cmm	2	2
$(3.4.6.4; 4.6.12)$	p6m	25	4

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