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The Inertia of the Hermitian H corresponding to H Unitary Matrices

Edna N. Gueco
University of the Philippines Baguio

Dennis I. Merino*
Southeastern Louisiana University

Agnes T. Paras
University of the Philippines Diliman

ABSTRACT

Let $H \in M_n(\mathbf{C})$ be Hermitian and nonsingular. An $A \in M_n(\mathbf{C})$ is called H unitary if $A^*HA = H$. The Jordan Canonical Form (JCF) of A is a direct sum of only two types: (i) $J_k(\lambda) \oplus J_k\left(\frac{1}{\overline{\lambda}}\right)$ with $|\lambda| > 1$ or (ii) $J_k(e^{i\theta})$ with $\theta \in \mathbf{R}$. If the JCF of A contains blocks of only type (i), then we show that $n = 2p$ is even and the inertia of H is $(p, p, 0)$. If the JCF of A is a single block of type (ii) and if $n = 2p$ is even, then we show that the inertia of H is $(p, p, 0)$. If the JCF of A is a single block of type (ii) and if $n = 2p + 1$ is odd, then we show that the inertia of H is either $(p + 1, p, 0)$ or $(p, p + 1, 0)$.

Keywords: Lorentz matrices, Λ_H orthogonal matrices

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LAYMAN'S ABSTRACT

Let $H \in M_n(\mathbf{C})$ be Hermitian and nonsingular. An $A \in M_n(\mathbf{C})$ is called H unitary if $A^*HA = H$. We study the inertia of H (number of positive and negative eigenvalues of H) in relation to the Jordan Canonical Form of A.

*Corresponding Author

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INTRODUCTION

The eigenvalues of a Hermitian matrix are real numbers. The set of all n-by-n Hermitian matrices includes the set of positive definite matrices (all the eigenvalues are positive), the set of all positive semidefinite matrices (all the eigenvalues are nonnegative), the negative semidefinite matrices (all the eigenvalues are nonpositive), and the negative definite matrices (all the eigenvalues are negative).

Let H be a Hermitian matrix, and let X be a nonsingular matrix that has the same size as H. Sylvester's Law of Inertia states that the number of positive eigenvalues, the number of negative eigenvalues, and the number of zero eigenvalues of H and $X * HX$ are the same [Theorem 4.5.8 of Horn and Johnson 2013]. Thus, if H is positive definite (respectively, positive semidefinite, negative semidefinite, negative definite), then $X * HX$ is positive definite (respectively, positive semidefinite, negative semidefinite, negative definite).

We let M_n(C) be the set of all n-by-n complex matrices. Let $H \in M_n$(C) be Hermitian. The *inertia* of H is the triple of integers (p,q,r), where p is the number of positive eigenvalues of H, q is the number of negative eigenvalues of H, and r is the number of zero eigenvalues of H.

A generalization of a unitary matrix ($U^*U = I$) is as follows. Let H_n be the set of all nonsingular Hermitian matrices in M_n(C), and let $H \in \mathrm{H}_n$(C) be given. An $A \in M_n$(C) is called *H unitary* if $A * HA = H$ [Mehrmann and Xu 1995]. If α is a nonzero real number and $H = \alpha I$, then an H unitary matrix is simply a unitary matrix. We let O_H be the set of all H unitary matrices in M_n(C).

Let $A \in M_n$(C) be nonsingular. Then, $A * HA = H$ if and only if $A(-H)A = -H$. Hence, $A \in \mathrm{O}_H$ if and only if $A \in \mathrm{O}_{-H}$. If $(p, q, 0)$ is the inertia of the nonsingular H, then $(p, q, 0)$ is the inertia of $-H$. Let $B \in M_n$(C) be similar to an element of O_H. We study the inertia of H. Notice that there are several H for which B is similar to an element of O_H.

There exist an integer $0 \le k \le n$ and a nonsingular $X \in M_n$(C) such that $X^{-*}HX^{-1} = L_k \equiv I_k \oplus -I_{n-k}$ [Theorem 4.5.8 of Horn and Johnson 2013]. Notice that the inertia of H is $(k, n-k, 0)$. An L_k unitary matrix is also called *Lorentz* (Autonne 1915; Givens 1940; Mehrmann and Xu 1999). One checks that $Q \in \mathrm{O}_H$ if and only if $XQX^{-1} \in \mathrm{O}_{Lk}$.

Let $A \in O_H$ be given. Then $H^{-1}A^*H = A^{-1}$. If A has a Jordan block corresponding to the eigenvalue λ, then A also has a Jordan block corresponding to the eigenvalue $\frac{1}{\overline{\lambda}}$. When $|\lambda| \neq 1$, these two blocks are different, and when $|\lambda| = 1$, these blocks are the same. In fact, a $B \in M_n(\mathrm{C})$ is similar to an element of O_H if and only if the Jordan Canonical Form of B can be expressed as a direct sum of only the following two types: (i) $J_p(\lambda) \oplus J_p\left(\frac{1}{\overline{\lambda}}\right)$ with $|\lambda| > 1$, or (ii) $J_p\left(e^{i\theta}\right)$ with $\theta \in \mathrm{R}$ [Proposition 4.3.3 of Gohberg and others 2000]. Most of the work in this area has been of this type: that is, for a given H, the properties of an $A \in O_H$ have been studied. Notice that if $A \in O_H$, then H is not unique. We investigate the possible inertia of H when the Jordan Canonical Form of B contains only type (i) or one block of type (ii). For these two cases, we determine all the possible inertia of H. For $n \leq 4$, we present the possible Jordan Canonical Form of $A \in O_H$ and the corresponding inertia of H.

We begin with the following observation, which can be easily proven.

Proposition 1. *Let $A \in M_n(\mathrm{C})$ be nonsingular. There exist an integer k and a $P \in O_{L_k}(\mathrm{C})$ such that A is similar to P if and only if there exists $H \in \mathrm{H}_n$ with inertia $(k, n-k, 0)$ such that $A^*HA = H$.*

Let B^k_{n-k} be the set of all $A \in M_n(\mathrm{C})$ such that there exists $H \in \mathrm{H}_n$ with inertia $(k, n-k, 0)$ and satisfies $A^*HA = H$. Notice that $I_n \in \mathrm{B}^k_{n-k}$ for each integer $k = 0, 1, \ldots, n$. Now, $A \in \mathrm{B}^k_{n-k}$ if and only if $A \in \mathrm{B}^{n-k}_k$.

Let $A \in \mathrm{B}^k_{n-k}$ be given. There exists $H \in \mathrm{H}_n$ with inertia $(k, n-k, 0)$ such that $A^*HA = H$. Let $S \in M_n(\mathrm{C})$ be nonsingular. Set $D = S^{-1}AS$, so that $A = SDS^{-1}$. We have $\left(SDS^{-1}\right)^* H \left(SDS^{-1}\right) = H$ and thus, $D^*\left(S^*HS\right)D = S^*HS$.

Proposition 2. *For each k, we have $\mathrm{B}^k_{n-k} = \mathrm{B}^{n-k}_k$. Let $A \in \mathrm{B}^k_{n-k}$ be given. If $S \in M_n(\mathrm{C})$ is nonsingular, then $S^{-1}AS \in \mathrm{B}^k_{n-k}$.*

Let $P \in O_{L_k} \subset M_n(\mathrm{C})$ and let $Q \in O_{L_j} \subset M_m(\mathrm{C})$ be given. Because $L_k \oplus L_j$ is similar to $L_{k+j} \in M_{n+m}(\mathrm{C})$, we have $P \oplus Q$ is similar to a matrix in $O_{L_{k+j}}$.

Proposition 3. *Let $A \in M_n(\mathrm{C})$ and let $B \in M_m(\mathrm{C})$ be given. Let $0 \leq k \leq n$ and $0 \leq j \leq m$ be integers. If A is similar to a matrix in O_{L_k} and if B is similar to a matrix in O_{L_j}, then $A \oplus B$ is similar to a matrix in $O_{L_{k+j}}$.*

MAIN RESULTS

Let $A \in M_n(\mathrm{C})$ be nonsingular. If λ is an eigenvalue of A, then $\overline{\left(\frac{1}{\lambda}\right)}$ is an eigenvalue of $\overline{A}^{-1}$. Notice that $\lambda = \overline{\left(\frac{1}{\lambda}\right)}$ if and only if $|\lambda| = 1$, that is, if and only if $\lambda = e^{i\theta}$ for some $\theta \in \mathrm{R}$. We let $\sigma(A)$ be the *spectrum* (set of eigenvalues) of A.

Let $A \in M_n(\mathrm{C})$ be nonsingular such that $\sigma(A) \cap \sigma\left(\overline{A}^{-1}\right) = \varnothing$. Set $B \equiv A \oplus \overline{A}^{-1}$, and consider the equation $B^* H = HB^{-1}$. Write $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ conformal to$B$. Then,

$$A^* H_{11} = H_{11} A^{-1}, \tag{1}$$

$$A^* H_{12} = H_{12} \overline{A}, \tag{2}$$

$$A^{-T} H_{21} = H_{21} A^{-1}, \tag{3}$$

and

$$A^{-T} H_{22} = H_{22} \overline{A}. \tag{4}$$

Since $\sigma(A^T) = \sigma(A)$, we have $\sigma(A^*) \cap \sigma(A^{-1}) = \varnothing = \sigma(A^{-T}) \cap \sigma(\overline{A})$. Theorem 4.4.6 in Horn and Johnson 1991 guarantees that $H_{11} = 0$ in equation (1) and that $H_{22} = 0$ in equation (4). Thus, any solution to $B^* HB = H$ has the form $H = \begin{bmatrix} 0 & H_{12} \\ H_{21} & 0 \end{bmatrix}$. If we require $H \in \mathrm{H}_{2n}$, then $H = \begin{bmatrix} 0 & H_{12} \\ H_{12}^* & 0 \end{bmatrix}$ and H_{12} is nonsingular. The inertia of such an H is $(n, n, 0)$ (Problem 40 in Horn and Johnson 2013).

Theorem 4. Let $A \in M_n(\mathrm{C})$ be nonsingular such that $\sigma(A) \cap \sigma\left(\overline{A}^{-1}\right) = \varnothing$. Then $A \oplus \overline{A}^{-1} \in \mathrm{B}_n^n$. If there is an integer k for which $A \oplus \overline{A}^{-1} \in \mathrm{B}_{2n-k}^k$, then $k = n$.

Let $B \in M_n(\mathrm{C})$ be given. Suppose that the JCF of B contains only Jordan blocks of type (i), say B is similar to

$$\left(J_{n_1}(\lambda_1) \oplus J_{n_1}\left(\overline{\frac{1}{\lambda_1}}\right)\right) \oplus \cdots \oplus \left(J_{n_p}(\lambda_p) \oplus J_{n_p}\left(\overline{\frac{1}{\lambda_p}}\right)\right)$$

with $|\lambda_j| > 1$ for each $j = 1, \ldots, p$. Let $k = n_1 + \cdots + n_p$. Then $m = 2k$. Let $C = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_p}(\lambda_p)$, and let $D = J_{n_1}\left(\overline{\frac{1}{\lambda_1}}\right) \oplus \cdots \oplus J_{n_p}\left(\overline{\frac{1}{\lambda_p}}\right)$.

Notice that D is similar to $\overline{C}^{-1}$, so that B is similar to $C \oplus \overline{C}^{-1}$. Notice also that $\sigma(C) \cap \sigma\left(\overline{C}^{-1}\right) = \varnothing$. Theorem 4 guarantees that $C \oplus \overline{C}^{-1} \in \mathrm{B}_k^k$. Proposition 2 guarantees that $B \in \mathrm{B}_k^k$. If there exists an integer j such that $B \in \mathrm{B}_{2k-j}^j$, then Proposition 2 guarantees that $C \oplus \overline{C}^{-1} \in \mathrm{B}_{2k-j}^j$. Theorem 4 guarantees that $j = k$.

Corollary 5. *Let $A \in M_n(\mathrm{C})$ be nonsingular. Suppose that there exist positive integers p, $n_1, \ldots, n_p$, nonzero $\lambda_1, \ldots, \lambda_p \in \mathrm{C}$ such that $|\lambda_i| > 1$ for each $i = 1, \ldots, p$ and A is similar to*

$$\left(J_{n_1}(\lambda_1) \oplus J_{n_1}\left(\overline{\frac{1}{\lambda_1}}\right)\right) \oplus \cdots \oplus \left(J_{n_p}(\lambda_p) \oplus J_{n_p}\left(\overline{\frac{1}{\lambda_p}}\right)\right).$$

Let $k = n_1 + \cdots + n_p$. Then $n = 2k$ and $A \in \mathrm{B}_k^k$. If there is an integer j for which $A \in \mathrm{B}_{2k-j}^j$, then $j = k$.

We now turn our attention to Jordan blocks of type (ii).

Let $A, B \in M_n(\mathrm{C})$ be given. Suppose that there exists a nonsingular X such that $A = XBX^{-1}$. Then, for every $H \in M_n(\mathrm{C})$, we have $A^* H A = H$ if and only if $B^*(X^* H X)B = (X^* H X)$. Now, the equation $A^* Z A = Z$ has a solution $H_A \in \mathrm{H}_n$ if and only if the equation $B^* Z B = Z$ has a solution $H_B \in \mathrm{H}_n$. Notice that H_B is a solution to $B^* Z B = Z$ if and only if $-H_B$ is also a solution.

Let $\theta \in \mathrm{R}$ be given. Then $J_n(e^{i\theta})$ is similar to $e^{i\theta} J_n(1)$. Now, notice that $A^* Z A = Z$ if and only if $(e^{i\theta} A)^* Z (e^{i\theta} A) = Z$. Thus, if

$$J_n^T(1) Z J_n(1) = Z \tag{5}$$

has a solution $H_1 \in \mathrm{H}_n$ with inertia $(k, n-k, 0)$, then

$$J_n^*(e^{i\theta}) Z J_n(e^{i\theta}) = Z \tag{6}$$

has a solution $H_2 \in \mathrm{H}_n$ with inertia $(k, n-k, 0)$ or $(n-k, k, 0)$. Conversely, if equation (6) has a solution $H_3 \in \mathrm{H}_n$ with inertia $(t, n-t, 0)$ then equation (5) has a solution $H_4 \in \mathrm{H}_n$ with inertia $(t, n-t, 0)$ or $(n-t, t, 0)$.

Let $Z = \lfloor z_{i,j} \rfloor$ be a solution to equation (5). Then, (see the proof of Lemma 4 in equation 6 of Horn and Merino 1999).

$$z_{i-1,\,j-1} + z_{i-1,\,j} + z_{i,\,j-1} = 0 \text{ for all } i,\, j \in \{1,\dots,\, n\} \quad (7)$$

where we adopt the convention that $z_{p,q} = 0$ if either $p = 0$ or $q = 0$. Set $j = 1$ and notice that $z_{i,1} = 0$ for $i = 1,\dots,\, n-1$. Set $j = 2$ and notice that $z_{i,2} = 0$ for $i = 1,\dots,\, n-2$. Moreover, we also have $z_{n-1,2} + z_{n,1} = 0$, so that $z_{n-1,2} = -z_{n,1}$. Continuing this process, we see that $z_{i,j} = 0$ whenever $i + j \le n$, and $z_{n-i,i+1} = (-1)^i z_{n,1}$.

Proposition 6. *Every solution to equation (5) has the form*

$$Z = \begin{bmatrix} 0 & & & 0 & (-1)^{n-1} z_{n,1} \\ & & 0 & & * \\ & 0 & z_{n,1} & * & \\ 0 & -z_{n,1} & * & & \\ z_{n,1} & * & & & * \end{bmatrix}. \quad (8)$$

If n is even and Z is Hermitian, then $z_{n,1}$ is purely imaginary. If n is odd and Z is Hermitian, then $z_{n,1}$ is real.

We show that equation (5) has a solution in H_n for each positive integer n. When n = 2k + 1 is odd, the proof of Lemma 3 of Horn and Merino 1999 (in equation 6, where *the* symmetric S in constructed) that Z may be taken to be Hermitian (in fact, real symmetric). For the case $n = 2\,k$ is even, we use mathematical induction.

For the base case, notice that $Z_2 = \begin{bmatrix} 0 & ia \\ -ia & b \end{bmatrix}$ with $a, b \in \mathrm{R}$ and $a \ne 0$ satisfies equation (5) and is Hermitian. Suppose that $Z_{2k} = \lfloor z_{i,j} \rfloor \in \mathrm{H}_{2k}$ satisfies equation (5) with $z_{1,2k} = ia$ and $0 \ne a \in \mathrm{R}$. Set

$$Z_{2k+2} = \begin{bmatrix} 0 & 0 & ia \\ 0 & -Z_{2k} & x \\ -ia & x^* & c \end{bmatrix}, \quad (9)$$

where $x \in \mathbb{C}^{2k}$ is to be determined. Write

$$J_{2k+2}(1) = \begin{bmatrix} 1 & e_1^T & 0 \\ 0 & J_{2k}(1) & e_{2k} \\ 0 & 0 & 1 \end{bmatrix}$$

conformal to Z_{2k+2}. Here, $e_i \in \mathbb{C}^{2k}$ is a vector with the i^{th} entry 1 and all other entries are 0. A calculation reveals that Z_{2k+2} is a solution to equation (5) if and only if the following equations are satisfied.

$$-J_{2n}^T(1)Z_{2k}e_{2k} + iae_1 + J_{2k}^T(1)x = x \quad (10)$$

and

$$e_{2k}^T x + x^* e_{2k} - e_{2k}^T Z_{2k} e_{2k} + c = c. \quad (11)$$

Equation (10) reduces to $J_{2n}^T(1)Z_{2k}e_{2k} - iae_1 = J_{2k}^T(0)x$, which has a solution since the first entry of the left hand side is 0. Every solution to equation (10) has a free variable x_{2k}. Equation (11) reduces to $2\,Re(x_{2k}) = e_{2k}^T Z_{2k} e_{2k} = e_{2k}^* Z_{2k} e_{2k}$, which has a solution since the right hand side is real because Z_{2k} is Hermitian.

We now look at the possible inertias of a nonsingular Hermitian matrix Z_n of the form (8). We begin with the case $n = 2k+1$ is odd. Suppose that $z_{n,1} = a > 0$. We use mathematical induction to show that Z_{2k+1} has $k+1$ positive eigenvalues and k negative eigenvalues. The base case is clear: $Z_1 = [a]$ has 1 positive eigenvalue (a) and 0 negative eigenvalue. Suppose that Z_{2p+1} has $p+1$ positive eigenvalues and p negative eigenvalues. Notice that

$$Z_{2p+3} = \begin{bmatrix} 0 & 0 & a \\ 0 & Z_{2p+1} & x \\ a & x^* & c \end{bmatrix}.$$

The eigenvalues of $\begin{bmatrix} 0 & 0 \\ 0 & Z_{2p+1} \end{bmatrix}$ are 0 together with $p+1$ positive eigenvalues and p negative eigenvalues of Z_{2p+1}. Cauchy's interlacing theorem for a bordered Hermitian matrix [Theorem 4.3.17 of Horn and Johnson 2013] now guarantees that Z_{2p+3} has $p+2$ positive eigenvalues and $p+1$ negative eigenvalues. If $z_{n,1} = a < 0$, then Z_{2k+1} has k positive eigenvalues and $k+1$ negative eigenvalues. A similar argument shows that Z_{2k+2} in equation (9) has $k+1$ positive eigenvalues and $k+1$ negative eigenvalues.

eigenvalues. A similar argument shows that Z_{2k+2} in equation (9) has $k+1$ positive eigenvalues and $k+1$ negative eigenvalues.

Proposition 7. Let n be a positive integer. Let $Z_n \in \mathrm{H}_n$ be a solution to equation (5). If $n = 2k$ is even, then the inertia of Z_n is $(k, k, 0)$. If $n = 2k+1$, then the inertia of Z_n is $(k+1, k, 0)$ or $(k, k+1, 0)$.

Let $P_n \in \mathrm{H}_n$ be a solution to equation (6). Suppose that the inertia of P_n is $(p, n-p, 0)$. Then equation (5) has a solution $T_n \in \mathrm{H}_n$ with inertia $(p, n-p, 0)$ or $(n-p, p, 0)$. If $n = 2k$ is even, then Proposition 7 guarantees that the inertia of T_n is $(k, k, 0)$. Hence, in this case, we have $p = k$. If $n = 2k+1$ is odd, then Proposition 7 guarantees that $p = k$ or $p = k+1$.

Theorem 8. *Let $\theta \in \mathrm{R}$ and let a positive integer n be given. If $n = 2k$ is even, then $J_n(e^{i\theta}) \in \mathrm{B}_k^k$. If there is an integer p for which $J_n(e^{i\theta}) \in \mathrm{B}_{n-p}^p$, then $p = k$. If $n = 2k+1$ is odd, then $J_n(e^{i\theta}) \in \mathrm{B}_{k+1}^k$ and $J_n(e^{i\theta}) \in \mathrm{B}_k^{k+1}$. If there is an integer p for which $J_n(e^{i\theta}) \in \mathrm{B}_{n-p}^p$, then $p = k$ or $p = k+1$.*

SMALL VALUES OF n

Let $A \in M_n(\mathrm{C})$ be H unitary, so that $A^* HA = H$. We look at the possible inertia of H when $n \le 4$. Collect all type (i) Jordan blocks of A in $B \in M_p(\mathrm{C})$ and collect all type (ii) Jordan blocks of A in $C \in M_{n-p}(\mathrm{C})$. Notice that $p = 2k$ is necessarily even and that $B \in \mathrm{B}_k^k$. Now, there exists a nonsingular $S \in M_n(\mathrm{C})$ such that $S^{-1}AS = B \oplus C$. Hence, we have $(B^* \oplus C^*)(S^* HS) = (S^* HS)(B^{-1} \oplus C^{-1})$. Write $S^* HS = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ conformal to $B \oplus C$. Notice that B^* and C^{-1} have disjoint spectra and C^* and B^{-1} also have disjoint spectra, so that $H_{12} = 0$ and $H_{21} = 0$.

Suppose that A is similar to a unitary matrix, say, XAX^{-1} is unitary. Then $A^*(X^* X)A = X^* X$. Notice that $X^* X$ is positive definite, so that the inertia of $X^* X$ is $(n, 0, 0)$. Conversely, suppose that H may be chosen so that the inertia of H is $(n, 0, 0)$. Then, such an H is positive definite. There exists a positive definite $P \in M_n(\mathrm{C})$ such that $H = P^2$ [Theorem 7.2.6 of Horn and Johnson 2013]. Now, $A^* HA = H$ becomes $A^* P^2 A = P^2$ and $(PAP^{-1})^*(PAP^{-1}) = I$, so that PAP^{-1} is unitary. If H has inertia $(n, 0, 0)$, then $-H$ has inertia $(0, n, 0)$.

Lemma 9. *Let $A \in M_n(\mathrm{C})$ be H unitary. Then A is similar to a unitary matrix if and only if H may be chosen so that the inertia of H is either $(n, 0, 0)$ or $(0, n, 0)$.*

Let $\theta_1,\ldots,\theta_n \in \mathbb{R}$ be given. If $A = \operatorname{diag}\left(e^{i\theta_1},\ldots,e^{i\theta_n}\right)$ and if $H = \operatorname{diag}(\pm 1,\ldots,\pm 1)$, then A is H unitary. The choice $H = I$ has the inertia $(n,0,0)$, as guaranteed by Lemma 9. Let $F \in M_n(\mathbb{C})$ be H unitary. If F is not diagonalizable or if F has an eigenvalue λ with $|\lambda| \neq 1$, then the inertia of H can neither be $(n,0,0)$ nor $(0,n,0)$.

Suppose that the distinct eigenvalues of C are $e^{i\theta_1},\ldots,e^{i\theta_k}$. There exist integers $m_1,\ldots,m_k$, nonsingular $T_2 \in M_{n-p}(\mathbb{C})$, and

$$C_j = J_{l_1}\left(e^{i\theta_j}\right)\oplus \cdots \oplus J_{l_{n_j}}\left(e^{i\theta_j}\right)$$

with $j = 1,\ldots,k$ such that $T_2^{-1}CT_2 = C_1 \oplus \cdots \oplus C_k$. In this case, $T_2^{*}H_{22}T_2$ is block diagonal, conformal to $T_2^{-1}CT_2$. The determination of the inertia of each block presents much difficulty.

If $n = 2$, then the only possible JCF of A are (1) $\operatorname{diag}\left(\lambda,\frac{1}{\overline{\lambda}}\right)$ with $|\lambda| > 1$, (2) $J_2\left(e^{i\theta}\right)$ with $\theta \in \mathbb{R}$, and (3) $\operatorname{diag}\left(e^{i\theta},e^{i\beta}\right)$ with $\theta,\beta \in \mathbb{R}$. In case (1), Theorem 4 guarantees that the only possible inertia of H is $(1,1,0)$. In case (2), Theorem 8 guarantees that the only possible inertia of H is $(1,1,0)$. For case (3), the possible inertias of H are $(2,0,0)$, $(1,1,0)$, and $(0,2,0)$.

If $n = 3$, then the only possible JCF of A are (1) $\operatorname{diag}\left(\lambda,\frac{1}{\overline{\lambda}},e^{i\theta}\right)$ with $|\lambda| > 1$

and $\theta \in \mathbb{R}$, (2) $J_2\left(e^{i\theta}\right)\oplus\left[e^{i\beta}\right]$ with $\theta,\beta \in \mathbb{R}$, (3) $J_3\left(e^{i\theta}\right)$ with $\theta \in \mathbb{R}$, and (4) $\operatorname{diag}\left(e^{i\theta},e^{i\beta},e^{i\delta}\right)$ with $\theta,\beta,\delta \in \mathbb{R}$.

In case (1), Corollary 5 and Theorem 8 guarantee that the possible inertias for H are $(2,1,0)$ and $(1,2,0)$. Notice that A is not similar to a unitary matrix, so that Lemma 9 guarantees that the inertia of H cannot be $(3,0,0)$ or $(0,3,0)$. In case (2), the only possible inertias for H are $(2,1,0)$ and $(1,2,0)$. The inertia of H cannot be $(3,0,0)$ or $(0,3,0)$ as A is not similar to a unitary matrix. For case (3), Theorem 8 guarantees that the inertia of H can only be $(2,1,0)$ or $(1,2,0)$. For case (4), the possible inertias of H are $(3,0,0)$, $(2,1,0)$, $(1,2,0)$, $(1,2,0)$ and $(0,3,0)$.

If $n = 4$, then the only possible JCF of A are (1) $J_2(\lambda)\oplus J_2\left(\frac{1}{\overline{\lambda}}\right)$ with $|\lambda| > 1$, (2) $\operatorname{diag}\left(\lambda,\frac{1}{\overline{\lambda}},\beta,\frac{1}{\overline{\beta}}\right)$ with $|\lambda|, |\beta| > 1$, (3) $\operatorname{diag}\left(\lambda,\frac{1}{\overline{\lambda}}\right)\oplus J_2\left(e^{i\theta}\right)$ with $|\lambda| > 1$ and $\theta \in \mathbb{R}$, (4) $\operatorname{diag}\left(\lambda,\frac{1}{\overline{\lambda}},e^{i\theta},e^{i\beta}\right)$ with $|\lambda| > 1$ and $\theta,\beta \in \mathbb{R}$, (5)

$J_2(e^{i\theta})\oplus J_2(e^{i\beta})$ with $\theta,\beta\in\mathrm{R}$, (6) $J_2(e^{i\theta})\oplus$ diag$(e^{i\beta},e^{i\delta})$ with $\theta,\beta,\delta\in\mathrm{R}$, (7) $J_3(e^{i\theta})\oplus[e^{i\beta}]$ with $\theta,\beta\in\mathrm{R}$, (8) $J_4(e^{i\theta})$ with $\theta\in\mathrm{R}$, and (9) diag$(e^{i\theta_1},e^{i\theta_2},e^{i\theta_3},e^{i\theta_4})$ with $\theta_1,\ldots,\theta_4\in\mathrm{R}$.

In case (1), Corollary 5 guarantees that the only possible inertia of H is (2,2,0). In case (2), Theorem 4 guarantees that the only possible inertia of H is (2,2,0). In case (3), the discussion at the beginning of this section shows that the only possible inertia of H is (2,2,0). In case (4), Corollary 5 and Theorem 8 guarantee that the possible inertias of H are (3,1,0), (2,2,0), and (1,3,0); and since A is not similar to a unitary matrix, Lemma 9 guarantees that H can neither have an inertia of (4,0,0) nor (0,4,0). Similarly, for case (6), the possible inertias of H are (3,1,0), (2,2,0), and (1,3,0); and neither (4,0,0) nor (0,4,0). In case (7), the possible inertias of H are (3,1,0), (2,2,0), and (1,3,0). In case (8), Theorem 8 guarantees that the only possible inertia of H is (2,2,0). In case (9), the possible inertias of H are (4,0,0), (3,1,0), (2,2,0), (1,3,0), and (0,4,0).

We now look at case (5). Suppose that $A=SBS^{-1}$, where $B=J_2(e^{i\theta})\oplus J_2(e^{i\beta})$ with $\theta,\beta\in\mathrm{R}$. If $\theta\neq\beta+2k\pi$ for any integer k, then $S^*HS=H_1\oplus H_2$, with $H_1,H_2\in\mathrm{H}_2$, each with inertia (1,1,0). Thus, the inertia of H is (2,2,0). If $\theta=\beta+2k\pi$ for some integer k, then $e^{i\theta}=e^{i\beta}$ and B is similar to $C=e^{i\theta}(J_2(1)\oplus J_2(1))$. Every Hermitian solution to $C^*ZC=Z$ has the form

$$Z=\begin{bmatrix}0 & ia & 0 & c\\ -ia & b & -c & d\\ 0 & -\bar{c} & 0 & ie\\ \bar{c} & \bar{d} & -ie & f\end{bmatrix}$$

with $a,b,e,f\in\mathrm{R}$. If Z is nonsingular, then a and c cannot be both 0. Now, the eigenvalues of $\begin{bmatrix}0 & ia & 0\\ -ia & b & -c\\ 0 & -\bar{c} & 0\end{bmatrix}$ are $\dfrac{b\pm\sqrt{b^2+4|c|^2+4a^2}}{2}$ and 0. Notice that

$$\frac{b-\sqrt{b^2+4|c|^2+4a^2}}{2}<0<\frac{b+\sqrt{b^2+4|c|^2+4a^2}}{2}.$$

Cauchy's interlacing theorem for a bordered Hermitian matrix [Theorem 4.3.17 of Horn and Johnson 2013] now guarantees that Z has 2 positive eigenvalues and 2 negative eigenvalues. This example shows the difficulty in handling the general case involving multiple Jordan blocks corresponding to the same eigenvalue ($e^{i\theta}$, with $\theta\in\mathrm{R}$).

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Dennis I. Merino <dmerino@selu.edu> is from the Southeastern Louisiana University in Hammond, LA.

Edna N. Gueco <edna.gueco@upb.edu.ph> is from the University of the Philippines in Baguio City.

Agnes T. Paras <agnes@math.upd.edu.ph> is from the University of the Philippines in Diliman, QC.