Science Diliman (January-June 2017) 29:1, 53-65

Effect of Spin Dimensionality in the Fidelity of Spin Chain Quantum Communication

Art Graeson B. Dumigpe*
University of the Philippines Diliman

Eric A. Galapon
University of the Philippines Diliman

ABSTRACT

We consider an unmodulated linear spin chain composed of two qubits at the ends, which undergo nearest-neighbor interactions, with an arbitrary spin between them. The state of the qubit on one end is to be transmitted through the arbitrary spin and is received by the qubit on the other end with some fidelity. We look at the behavior of the average fidelity of state transfer through time as affected by the spin quantum number of the arbitrary spin in between the qubits. We find that the higher the spin quantum number becomes, the earlier it takes to achieve perfect or nearly perfect state transfer. Moreover, when the channel with arbitrary spin is subjected to environment interaction, results from the calculation of average fidelity suggest that increasing the spin quantum number of the channel provides a countermeasure to the effects of decoherence induced by the environment.

Keywords: Quantum communications, spin chain, fidelity, open quantum system

INTRODUCTION

In quantum information and computation, it is often a very important task to allow quantum states to be transferred from one place to another. Although there is an existing protocol for quantum state transfer, wherein a mobile carrier is sent away to a distant location (Gisin and Thew 2007), it is practical to devise a scheme suitable for short distances without requiring modulation on the quantum system.

Corresponding Author

ISSN 0115-7809 Print / ISSN 2012-0818 Online

Bose (2003) proposed the use of a chain of qubits or quantum objects with two degrees of freedom to carry out such task. The system does not require any interfacing at all because the interaction between the qubits facilitates the quantum communication process itself (Bose 2003, 2007). He found out that, for 80 spins, the fidelity (or probability of successful state transfer) of this protocol surpasses 2/3, which is the maximum of that of local operations and classical communication (LOCC) (Horodecki et al. 1999). This study paved way to the idea of designing quantum wires for connecting quantum computers.

Most of the literature in line with the study of spin chains for quantum communication involves qubits. Different cases were considered: whether the chain resembles a ferromagnet (Park et al. 2012) or an anti-ferromagnet (Ye et al. 2003); whether the interaction is long-ranged or nearest-neighbor (Zenchuk 2012); the different geometries of the spin chain (Bugarth and Bose 2005); and, whether the chain is free from environment interaction or not (Alvarez et al. 2010). Although there is existing work on such systems with higher spins (Moukouri 2006), only homogeneous chains were given much attention.

In this work, we consider a linear open-ended 3-component spin chain, wherein the ends are qubits and the channel spin object has an arbitrary spin. We evaluate the fidelity of quantum communication through this chain as we increase the spin of the channel, thus relating the spin value of a single-object channel to its capacity to transmit a quantum state efficiently. Moreover, this study can provide insights on the ways to build compact quantum computers, and have an impact on spintronics (electronics concerning higher spins).

PRELIMINARIES

Consider a linear spin chain with N spins subject to an external field. Given that the interactions are of the nearest-neighbor type, the associated Heisenberg XY Hamiltonian is given by Rico et al. (2004):

$$H = -\frac{\hbar}{2}\sum_{i=1}^{N-1}\left(\frac{1+\gamma}{2}S_i^x S_{i+1}^x + \frac{1-\gamma}{2}S_i^y S_{i+1}^y\right) - \frac{\hbar}{2}\sum_{i=1}^{N}\lambda S_i^z, \tag{1}$$

where the operators $S^i\,(i = x, y, z)$ are the spin operators with the number of matrix elements depending on the spin quantum number s of the object (i.e., $d = 2s + 1$). Moreover, s admits positive integer and half-integer values (e.g., 0, 1/2, 1, 3/2, 2, etc.). The subscripts i, $i+1$ in the first summation indicate that the interaction between the spins is nearest-neighbor, and the parameter γ describes the anisotropy of such interaction. The second summation indicates the individual contributions of the spins due to their angular frequency described by λ, which is proportional to the magnetic field strength of one component on the chain.

In this paper, we consider three components for our spin chain, wherein the first component is the sender and the last is the receiver, which are both qubits or two-level systems. Between them is the channel spin with associated quantum number *s*. In this manner, we can rewrite the Hamiltonian as

$$\begin{aligned} H = &-\frac{1}{2}\left(\frac{1+\gamma}{2}\left(S_{1/2}^{x}\otimes S_{s}^{x}\otimes \mathbb{I}_{2}+\mathbb{I}_{2}\otimes S_{s}^{x}\otimes S_{1/2}^{x}\right)\right)\\ &-\frac{1}{2}\left(\frac{1-\gamma}{2}\left(S_{1/2}^{y}\otimes S_{s}^{y}\otimes \mathbb{I}_{2}+\mathbb{I}_{2}\otimes S_{s}^{y}\otimes S_{1/2}^{y}\right)\right)\\ &-\frac{1}{2}\left(\lambda\left(S_{1/2}^{z}\otimes \mathbb{I}_{d}\otimes \mathbb{I}_{2}+\mathbb{I}_{2}\otimes S_{s}^{z}\otimes \mathbb{I}_{2}+\mathbb{I}_{2}\otimes \mathbb{I}_{d}\otimes S_{1/2}^{z}\right)\right), \end{aligned} \tag{2}$$

where we introduced the notion of tensor products, which best describes multipartite quantum systems. In this Hamiltonian, the subscripts in the spin operators indicate the spin of the state they operate on. The identity operator $\mathbb{I}$ denotes that the state remains the same, with the subscript describing the size of the matrix.

We evaluate the dynamics of the spin chain communication protocol by solving for the time-evolved state of the entire system with the dynamics governed by the Hamiltonian. For the time-evolution of the overall state where environment interaction is not present, we use the time-dependent Schrödinger equation given by

$$i\hbar \frac{\partial}{\partial t}\left|\Psi\right\rangle = H\left|\Psi\right\rangle, \tag{3}$$

where the overall state representation $\left|\Psi\right\rangle$ works for pure states. In general, we admit mixed states represented by density operators, so that the equation of motion to be used is the von Neumann equation given by

$$\frac{\partial}{\partial t}\rho = -\frac{i}{\hbar}[H,\rho], \tag{4}$$

where ρ is the density operator representation of the overall state, and $[H,\rho]$ is the commutator evaluated as $H\rho - \rho H$. Equations 3 and 4 suggest unitary evolution of the states.

In the case wherein environment interaction is present, and thus, wherein the evolution is not unitary when Equations 3 and 4 are used, the differential equation useful to describe the time-evolution is the Lindblad master equation given by

$$\frac{\partial}{\partial t}\rho = \frac{\hbar}{i}[H,\rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right), \tag{5}$$

where the L_k's are the Lindblad operators describing the effect of the environment on the system, and the operation $\{L_k^\dagger L_k, \rho\}$ is the anticommutator evaluated as $L_k^\dagger L_k \rho + \rho L_k^\dagger L_k$.

The next task is to extract information on the final state accessed by the receiver and compare it with the sender's initial state. We do this by *partial tracing* over the subsystems other than the subsystem of interest. If we label the sender subsystem as *A*, channel as *C*, and the receiver as *B*, the final state ρ_B of subsystem *B* is given by

$$\rho_B = Tr_{AC}\,\rho = \sum_{n=1}^{2} \sum_{m=1}^{d} \langle n | \langle m | \rho | m \rangle | n \rangle, \tag{6}$$

where $|n\rangle$ and $|m\rangle$ provide the basis states for subsystem *A* and *C*, respectively. We then evaluate the quantum communication protocol with a figure of merit called the fidelity. This measures the probability of the final state of *B* being identical to the initial state of *A*. If we initialize the state of *A* to be a pure state, the fidelity is given by

$$F = \langle \psi_A | \rho_B | \psi_A \rangle. \tag{7}$$

Spin Chain Communication Protocol

The quantum communication scheme presented in this paper can be visualized using Figure 1. Alice and Bob control each of the respective ends of the chain, and Alice's task is to transfer the state of the qubit she controls, and Bob has to measure that state at an indefinite time. Alice sends an unknown initial state to the receiver through a chain that starts with ground states (i.e., object *C* and qubit *B* are in $|0\rangle$).

The overall initial state is written as

$$|\Psi\rangle = |\psi_A\rangle \otimes |0_C\rangle \otimes |0_B\rangle. \tag{8}$$

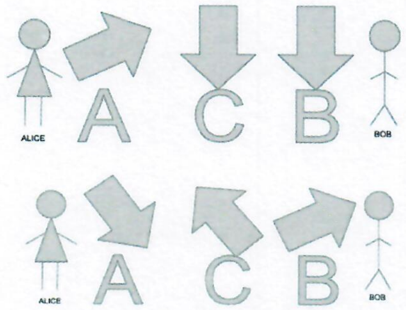

Figure 1. Diagrammatic display of the quantum communication protocol composed of quantum subsystems A (controlled by Alice), B (controlled by Bob), and C (top). Alice sends an unknown state through the chain where C and B are in the $|0\rangle$state. Bob expects to retrieve the state sent by Alice (bottom).

We introduce the Bloch sphere, known as the geometric representation of the state space of pure states for qubits. In this manner, we describe our unknown state in terms of angular quantities θ and ϕ, and the expression is given by

$$|\psi_A\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle. \tag{9}$$

The overall initial state is then written as

$$|\Psi\rangle = \cos\frac{\theta}{2}|000\rangle + e^{i\phi}\sin\frac{\theta}{2}|100\rangle. \tag{10}$$

Since the initial state is unknown, the equation for fidelity will be transformed into an integral over all angular values in the Bloch sphere, denoted as the average fidelity, given by

$$F_{ave} = \frac{\int F d\Omega}{\int d\Omega}, \tag{11}$$

where $d\Omega$ is the differential solid angle, and the integration is done over all directions. We derive formal expressions for the average fidelity of spin chain communication from qubit A to qubit B for cases where the chain is isolated and where the chain is open to environment interaction.

RESULTS AND DISCUSSION

Average Fidelity for the Isolated Spin Chain

We let the initial state in Equation 10 evolve unitarily by way of the Schrödinger equation given in Equation 3. The solution is given by

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle, \tag{12}$$

where $U(t) = e^{-iHt/\hbar}$ is the unitary operator. We insert identity operators in terms of the entire spin chain system's basis vectors, given as

$$\mathbb{I} = \sum_{k=0}^{1}\sum_{l=0}^{d-1}\left(|kl0\rangle\langle kl0| + |kl1\rangle\langle kl1|\right), \tag{13}$$

where k labels the basis states of $|\psi\rangle_A$, l labels the basis states of the arbitrary spin C, and 0 and 1 labels the basis states of Bob. We now end up with the following expressions for the final state $|\Psi(t)\rangle$:

$$|\Psi(t)\rangle = \left(\sum_{k=0}^{1}\sum_{l=0}^{d-1}\left(|kl0\rangle\langle kl0| + |kl1\rangle\langle kl1|\right)\right)U(t)|\Psi(0)\rangle \tag{14}$$

$$= \sum_{k=0}^{1}\sum_{l=0}^{d-1}\left(\langle kl0|U(t)|\Psi(0)\rangle|kl0\rangle + \langle kl1|U(t)|\Psi(0)\rangle|kl1\rangle\right). \tag{15}$$

We then retrieve the receiver state ρ_B at anytime t by doing the partial trace operation given in Equation 6. By writing the time-evolved state as a density matrix, (i.e., $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$), we have the receiver state

$$\rho_B(t) = \sum_{k=0}^{1}\langle k|\left(\sum_{l=0}^{d-1}\langle l|\rho(t)|l\rangle\right)|k\rangle. \tag{16}$$

Introducing Kraus operators *M(t)*, which maps an input state into an output state, we can expand Equation 16 in terms of the sender states. Defining our Kraus operators to be

$$M_{kl}(t) = \begin{pmatrix} g_{000}^{kl0}(t) & g_{100}^{kl0}(t) \\ g_{000}^{kl1}(t) & g_{100}^{kl1}(t) \end{pmatrix}, \tag{17}$$

where the elements are transition amplitudes

$$g_x^y(t) = \langle y|U(t)|x\rangle \tag{18}$$

indicating transition from a state $|x\rangle$ to a state $|y\rangle$. We express our final receiver state as

$$\rho_B(t) = \sum_{k=0}^{1}\sum_{l=0}^{d-1} M_{kl}(t)\rho_A M_{kl}^{\dagger}(t). \tag{19}$$

Finally, we evaluate the average fidelity of quantum communication through this spin chain. By solving equations 7 and 11, we are left with the following expression, purely in terms of transition amplitudes:

$$F_{ave} = \frac{1}{2}\sum_{k=0}^{1}\sum_{l=0}^{d-1}\left[\frac{2}{3}(|g_{000}^{kl0}|^2 + |g_{100}^{kl1}|^2) + \frac{1}{3}(|g_{100}^{kl0}|^2 + |g_{000}^{kl1}|^2 + g_{100}^{kl1}g_{000}^{kl0*} + g_{100}^{kl1*}g_{000}^{kl0})\right]. \tag{20}$$

With $d = 2s+1$ corresponding to the spin of the quantum object *C*, the sum in Equation 20 has $2d$ terms. We solve $8d$ amplitudes overall for each value of spin *s* we consider.

Average Fidelity for the Open Chain

We now derive the average fidelity for the case wherein the spin chain is coupled to an external environment. The state evolution is governed by the Lindblad master equation in (5). We use only one Lindblad operator to represent environment action onto the system, and we have it in the following form:

$$L = \mathbb{I}_2 \otimes l \otimes \mathbb{I}_2, \tag{21}$$

where the environment acts on the channel spin *C* through the operator *l* (with dimension d), and trivially, on the qubits *A* and *B*.

We choose the *l*-operators to be in the form of raising and lowering operators, with the spin dimensions of the channel spin written as

$$l_+ = \alpha_+ S_+, \tag{22}$$

$$l_- = \alpha_- S_-, \tag{23}$$

where $S_+ = S_x + iS_y$ and $S_- = S_x - iS_y$. The factors α_+ and α_- describe the coupling strength between the environment and the system, and they were taken to be constants. This choice of Lindblad operators is motivated by the interaction of a two-level atom with an external electromagnetic field. The resulting dissipative master equation is of Lindblad type and contains σ_+ and σ_-, which are raising and lowering operators for two-level systems (Breuer and Petruccione 2002). The

Lindblad operators and the Hamiltonian are time-independent, and thus, we can start by condensing Equation 5 into the expression

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\rho, \tag{24}$$

where $\mathcal{L}$ is the linear operator describing the differentiation with respect to time of the density operator. The solution takes the form

$$\rho(t) = e^{t\mathcal{L}}\rho(0). \tag{25}$$

With $\rho(0) = |\Psi(0)\rangle\langle\Psi(0)|$, we now write the final state as

$$\begin{aligned}\rho(t) &= \cos^2\left(\frac{\theta}{2}\right)G_{000}^{000}(t) + e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)G_{000}^{100}(t)\\ &+e^{i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)G_{100}^{000}(t) + \sin^2\left(\frac{\theta}{2}\right)G_{100}^{100}(t),\end{aligned} \tag{26}$$

with

$$G_x^y(t) = e^{t\mathcal{L}}|x\rangle\langle y| \tag{27}$$

being operators themselves.

We then extract the receiver state at any time t by partial tracing Equation 26, and we get

$$\begin{aligned}\rho_B(t) = \sum_{m=1}^{0}\sum_{l=0}^{d-1}\Big(&\cos^2\left(\frac{\theta}{2}\right)\langle ml\,|\,G_{000}^{000}\,|\,lm\rangle + e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\langle ml\,|\,G_{000}^{100}\,|\,lm\rangle\\ &+e^{i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\langle ml\,|\,G_{100}^{000}\,|\,lm\rangle + \sin^2\left(\frac{\theta}{2}\right)\langle ml\,|\,G_{100}^{100}\,|\,lm\rangle\Big).\end{aligned} \tag{28}$$

Finally, we write the expression for the average fidelity for any time t by solving Equations 7 and 11, and this yields

$$\begin{aligned}F_{ave} = \frac{1}{2}\sum_{m=1}^{0}\sum_{l=0}^{d-1}\Big[&\frac{2}{3}\Big(\langle 0ml\,|\,G_{000}^{000}\,|\,lm0\rangle + \langle 1ml\,|\,G_{100}^{100}\,|\,lm1\rangle\Big)\\ &+\frac{1}{3}\Big(\langle 1ml\,|\,G_{000}^{000}\,|\,lm1\rangle + \langle 0ml\,|\,G_{000}^{100}\,|\,lm1\rangle\\ &+\langle 1ml\,|\,G_{100}^{000}\,|\,lm0\rangle + \langle 0ml\,|\,G_{100}^{100}\,|\,lm0\rangle\Big)\Big].\end{aligned} \tag{29}$$

Similar to that of Equation 20, Equation 29 has $2d$ terms. Here, we derive $8d$ operators in the form of Equation 27 and evaluate $12d$ transition amplitudes for the average fidelity given a spin number s.

From the average fidelity expressions in Equations 20 and 29, we now look at the effects of using a higher-spin channel in the average fidelity of quantum communication with respect to time. We employed numerical methods to carry out such task, as the expressions contain transition amplitude terms which are rather difficult to evaluate by hand. We used MATLAB R2013b installed on a computer with an Intel® Core™ i7-4790 CPU @ 3.60 GHz processor with 64 GB of RAM for the numerical computations and for displaying results in graphical and tabular form.

We first look at the cases where we have our quantum system to be isolated (i.e., we solve for the average fidelity given in Equation 20). We plot the results by considering spin *s* values of up to *s*=5/2 for the following parameter pairs {γ,λ} of the Hamiltonian, namely {0,10}, {1,10}, and {0,5} (Figure 2). As seen in Figures 2a and 2b, the general behavior of the fidelity plots do not differ much, except for a small time interval. In the case where the Hamiltonian parameter pair is {0,5}, if we

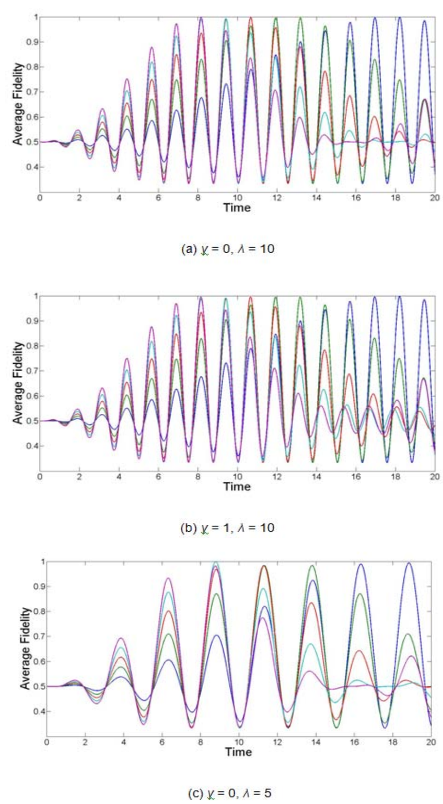

Figure 2. Average fidelity plots for quantum channels with s = 1/2 (blue), s = 1 (green), s = 3/2 (red), s = 2 (cyan), and s = 5/2 (magenta).

compare with the previous plots, the frequency of attaining maximum values of the average fidelity was observed to decrease. We attribute this effect to the magnetic field strength of the spin chain probing the transfer of states from Alice to Bob.

We now look at the direct effect of the spin *s* of the channel to the average fidelity of the quantum communication through our spin chain. In Figure 2, the plots representing the different spins considered are shown in different colors. One of the recurring trends is that the graph representing the highest spin value of $s=5/2$ tends to reach the target maximum average fidelity first, which is ideally 1 or a value very close to it. Table 1 displays the points in the time axis where the first maximum average fidelity happens for each chain with channel spin *s*. This implies that the spin quantum number of the channel positively affects the communication protocol (i.e., it probes the state of the receiver to mimic the state of the sender at an earlier time), and this works on average for an arbitrary initial state. The other recurring trend is that, for certain points in the time axis before those corresponding to the average fidelity of 1, the plots with the higher spin *s* tend to have the highest local maxima and the lowest local minima. We attribute this to the oscillatory behavior of the time evolution of the system reflected in the final receiver state. We also note that the resulting minimum value of the average fidelity for all plots considered is 1/3. We say that, in this quantum communication protocol, the channel with higher spin gives more efficiency to the state transfer, since the results provide a computable time of achieving maximum fidelity. Therefore, in this time, the receiver state can be measured before the spin chain evolves, and thus, the fidelity goes to a minimum.

Finally, we see the effect of environment interaction on the average fidelity of state transfer for this case. Using the Lindblad operators described in Equations 21, 22, and 23, we only focus on the cases where the interaction constants α_+ (Figure 3) and α_- (Figure 4) are set to 1. A noticeable feature of the plots is that those corresponding to the different spin values tend to overlap after a certain amount of time. The second trend attributed to different spin *s* for the isolated system case can be observed in the open system cases by zooming in on the plots to a certain

Table 1. Time (in units of λ^{-1}) it takes for each channel spin to attain first maximum fidelity.

Spin s	½	1	3/2	2	5/2
$\gamma = 0, \lambda = 10$	18.22	11.94	10.68	9.423	8.167
$\gamma = 1, \lambda = 10$	18.22	11.94	10.68	9.423	8.167
$\gamma = 0, \lambda = 5$	18.85	13.81	11.34	8.813	8.781

extent of time. For both cases, we can see that, for a short period of time, the effect shown in the isolated case still manifests in the open system cases we considered. These results suggest that placing a high-spin quantum object as a channel between the two qubits can counteract the effects of decoherence.

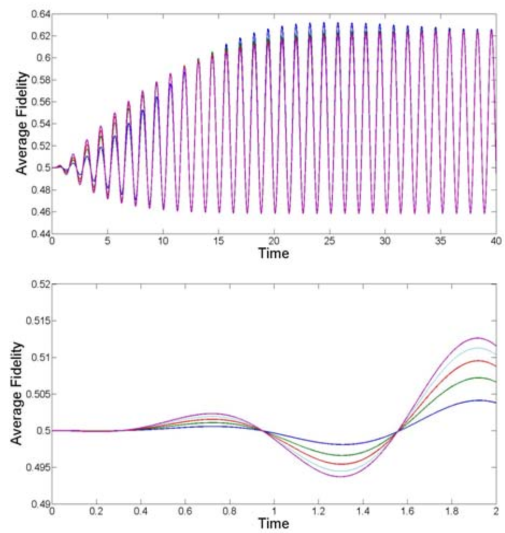

Figure 3. Average fidelity plots for quantum channels with s = 1/2 (blue), s = 1 (green), s = 3/2 (red), s = 2 (cyan), and s = 5/2 (magenta). Hamiltonian parameters are $\gamma = 0$, $\lambda = 10$, and $l = \alpha_+ S_+$.

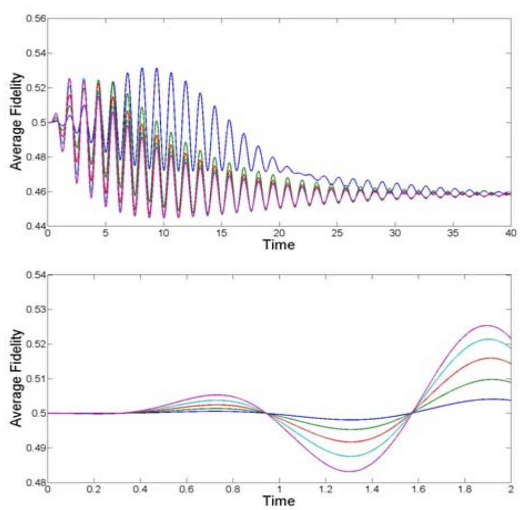

Figure 4. Average fidelity plots for quantum channels with s = 1/2 (blue), s = 1 (green), s = 3/2 (red), s = 2 (cyan) and s = 5/2 (magenta). Hamiltonian parameters are $\gamma = 0$, $\lambda = 10$, and $l = \alpha_- S_-$

CONCLUSIONS AND RECOMMENDATIONS

We have indeed shown that, for a linear and open-ended 3-component quantum spin chain, the spin of the channel affects the efficiency of quantum communication (i.e., we achieve nearly perfect to perfect state transfer at an earlier time as the spin quantum number of the channel goes higher). Furthermore, we have seen that, for up to spin s = 5/2, this effect still manifests in the presence of environment interaction. We have shown that using a higher spin value for the channel keeps the efficiency of our quantum communication scheme high for a short period of time when it is subject to the detrimental decoherence induced by environment interaction.

Extensions of this work include displaying results for even higher spins, in order to show that increasing the spin of this channel can indeed counter the environment-induced loss of information. Addition of another component in the channel with varying spin can also be considered.

ACKNOWLEDGMENTS

This work was funded by the Office of the Vice President for Research and Development through the OVCRD-Outright Research Grant No. 141408 PNSE.

REFERENCES

Alvarez G, Ajoy A, Peng X, Suter D. 2010. Performance comparison of dynamical decoupling sequences for a qubit in a rapidly fluctuating spin bath. Physical Review A. 82(4):042306.

Bose S. 2003. Quantum communication through an unmodulated spin chain. Physical Review Letters. 91(20):207901.

Bose S. 2007. Quantum communication through spin chain dynamics: An introductory overview. Contemporary Physics. 48(1):13-30.

Breuer H, Petruccione F. 2002.The theory of open quantum systems. New York (NY): Oxford University Press.

Bugarth D, Bose S. 2005. Conclusive and arbitrarily perfect quantum state-transfer using parallel spin-chain channels. Physical Review A. 71(5):052315.

Gisin N, Thew R. 2007. Quantum communication. Nature Photonics. 1(3):165-171.

Horodecki P, Horodecki M, Horodecki R.1999. General teleportation channel, singlet fraction, and quasidistillation. Physical Review A. 60(3):1888-1898.

Moukouri S. 2006. Neel and disordered phases of coupled Heisenberg chains with S = 1/2 to S = 4. Journal of Statistical Mechanics: Theory and Experiment. 2006(2):P02002.

Park H, Um J, Hinrichsen H. 2012. Entanglement versus mutual information in quantum spin chains. Journal of Statistical Mechanics. 10(2012):P10026.

Rico E, Latorre JI, Vidal G. 2004. Ground state entanglement in quantum spin chains. Quantum Information and Computation. 4(1):48-92.

Ye F, Ding GH, Xu BW. 2003. Numerical study of alternating quantum spin-1/2 chains. Physica A. 318(3-4):414-422.

Zenchuck A. 2012. Information propagation in a quantum system: examples of open spin-1/2 chains. Journal of Physics A Mathematical and Theoretical 45(11):115306.

Art Graeson B. Dumigpe <agdumigpe@nip.upd.edu.ph> is an MS Physics student at the University of the Philippines Diliman currently on his first year. He is enrolled as a scholar of the AST-HRDP. He works under the supervision of Dr. Eric Galapon.

Eric A. Galapon <eagalapon@up.edu.ph> is a Professor at the National Institute of Physics, University of the Philippines Diliman. His fields of interest include foundations of quantum mechanics, quantum measurement theory, quantum time problem, and mathematical physics.