The Jordan Canonical Form of a Product of Elementary S-unitary Matrices

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ABSTRACT

Let $S$ be an $n$-by-$n$, nonsingular, and Hermitian matrix. A square complex matrix $Q$ is said to be $S$-unitary if $Q^*SQ = S$. An $S$-unitary matrix $Q$ is said to be elementary if rank$(Q - I) = 1$. It is known what form every elementary $S$-unitary can take, and that every $S$-unitary can be written as a product of elementary $S$-unitaries. In this paper, we determine the Jordan canonical form of a product of two elementary $S$-unitaries.

Keywords: elementary $S$-unitary matrix, Hermitian matrix, Jordan canonical form

INTRODUCTION

Let $M_n$ be the set of all $n$-by-$n$ matrices with entries in the complex field $\mathbb{C}$ and let $GL_n$ be the set of all nonsingular matrices in $M_n$. Let $S \in GL_n$ be Hermitian. A $Q \in M_n$ is said to be $S$-unitary if $Q^*SQ = S$, where $Q^*$ is the conjugate transpose of $Q$ (Gohberg et al. 2005). If $S = I$, then the set of $S$-unitary matrices in $GL_n$ coincides with the set of unitary matrices. Let $U_S$ be the set of all $S$-unitary matrices. Observe that $U_S$ is nonempty since $I \in U_S$. If $Q \in U_S$, then $Q^{-1}$ is similar to $Q^*$, $|\det Q| = 1$, and $\alpha Q \in U_S$ for all $\alpha \in \mathbb{C}$ with modulus 1. Moreover, $U_S$ is a group under multiplication and consists of all matrices in $M_n$ that preserve the scalar product $[u,v]_S = u^*Sv$ for all $u, v \in \mathbb{C}^n$.

An $H \in U_S$ is called elementary if rank$(H - I) = 1$. Let $H_S$ be the set of all elementary $S$-unitary matrices. When $S$ is Hermitian, $H_S = K_S \cup L_S$, where

$$K_S = \{K_{x,r} = I + ixx^*S : x \in \mathbb{C}^n \setminus \{0\}, x^*Sx = 0, \text{ and } r \in \mathbb{R} \setminus \{0\}\}$$

and

$$L_S = \{L_{x,\phi} = I + \frac{(e^{i\phi} - 1)}{x^*Sx}xx^*S : x \in \mathbb{C}^n, x^*Sx \neq 0, \phi \in \mathbb{R}, \text{ and } e^{i\phi} \neq 1\}$$

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(Catbagan 2015). If \( v \in \mathbb{C}^n \) such that \( v^*Sv \neq 0 \), the \( \Lambda_x \)-Householder matrix \( Sv = I - \frac{2}{v^*v}vv^*S \) generalizes the Householder matrix of \( v \), which is equal to \( L_{v,v} \) for \( S = I \) (Merino et al. 2011; Horn and Johnson 2013). If \( K_{x_1} \in K_n \), then \( K_{x_1}^{-1} = K_{x_1} \) and \( K_{x_1} \) is similar to \( I_{n-2} \oplus J_2 \) (1). If \( L_{x_1} \in L_n \), then \( L_{x_1}^{-1} = L_{x_1} \), and \( L_{x_1} \) is similar to \( I_{n-1} \oplus [e^{i\varphi}] \). Hence \( H \in H_n \) if and only if \( H^{-1} \in H_n \). Thus, \( I \) is a product of two elements of \( H_n \). Moreover, if \( A \in U_n \), then \( A \) can be written as a product of elements of \( H_n \) (Catbagan 2015). Thus, the elements of \( H_n \) generate the group \( U_n \). Since there are two types of elements of \( H_n \), there are three types of products of two elements of \( H_n \) up to similarity. We wish to determine which Jordan canonical forms arise for each possibility, since the Jordan canonical form of a matrix reveals a lot of information such as its rank, nullity, eigenvalues, and their algebraic and geometric multiplicities. Analogous results for symplectic matrices and \( J \)-Householder matrices can be found in de la Rosa et al. (2012).

**PRELIMINARIES**

If \( S = P^*P \) for some \( P \in GL_n \), then \( x^*Sx > 0 \), when \( 0 \neq x \in \mathbb{C}^n \); and \( Q \in U_n \) if and only if \( PQP^{-1} \in U_n \). Hence when \( S \) is positive definite, \( H_n = L_n \), and every \( S \)-unitary is similar to a unitary matrix. Since \( U_n = U_{2n} \), from now on we only consider \( S \) that is \( * \)-congruent to \( I_n \oplus -I_{n-k} \) for \( 0 < k < n \), that is \( S = P^* (I_n \oplus -I_{n-k}) P \), for some \( P \in GL_{2n} \).

Let \( n \) be a positive integer such that \( n \geq 2 \), and \( T \subseteq \mathbb{C}^n \) be nonempty. Let \( T^S \) be the \( S \)-perpendicular subspace of \( T \) defined by

\[
T^S = \{ z \in \mathbb{C}^n \mid x^*Sz = 0, \text{ for all } x \in T \}.
\]

Then \( \dim T^S = n - \dim(\text{span} T) \), since \( T^S = (S(\text{span} T))^\perp \), which is the orthogonal complement of \( S(\text{span} T) \) with respect to the usual inner product on \( \mathbb{C}^n \), and \( \mathbb{C}^n = W \oplus W^\perp \) for any subspace \( W \) of \( \mathbb{C}^n \). Let \( H_x, H_y \in H_n \) and \( A = H_x H_y \). Then \( H_x = I + \alpha xx^*S \) and \( H_y = I + \beta yy^*S \), for some nonzero \( \alpha, \beta \in \mathbb{C} \). If \( \{x, y\} \) is linearly dependent, then \( y = \delta x \), for some \( \delta \in \mathbb{C} \). This implies

\[
A = I + \alpha xx^*S + \beta yy^*S + \alpha \beta xx^*Syy^*S = I + (\alpha + \beta |\delta|^2 + \alpha \beta |\delta|^2) xx^*S.
\]

Hence \( A = I + \mu xx^*S \), where \( \mu = \alpha + \beta |\delta|^2 + \alpha \beta |\delta|^2 \). If \( \mu = 0 \), then \( A = I \), which implies \( H_x = H_y^{-1} \). If \( \mu \neq 0 \), then \( \text{rank}(A - I) = 1 \), and since \( A \in U_n \), we have \( A \in H_n \).
Suppose \( \{x, y\} \) is linearly independent. Let \( z \in \mathbb{C}^n \) be given. Suppose \( z \in \ker(A - I) \), that is, \( Az = z \). Then

\[
0 = (A - I)z = (\alpha x^*Sz)x + (\beta y^*Sz)y + \alpha\beta(x^*Sy)(y^*Sz)x.
\]

Since \( \{x, y\} \) is linearly independent, and \( \alpha, \beta \) are nonzero, we have \( y^*Sz = 0 \) and it follows that \( x^*Sz = 0 \). Conversely, suppose \( z \in \{x, y\}^5 \). Then \( x^*Sz = y^*Sz = 0 \) and so

\[
(A - I)z = \alpha(x^*Sz)x + \beta(y^*Sz)y + \alpha\beta(x^*Sy)(y^*Sz)x = 0,
\]

that is, \( z \in \ker(A - I) \). Therefore \( \ker(A - I) = \{x, y\}^5 \).

**Lemma 1.** Let \( S \in \text{GL}_n \) be Hermitian and let \( x, y \in \mathbb{C}^n \) be nonzero. Suppose \( H_x, H_y \in H_S \) and \( A = H_x H_y \). If \( \{x, y\} \) is linearly dependent, then \( A = I \) or \( A \in H_S \). If \( \{x, y\} \) is linearly independent, then \( \ker(A - I) = \{x, y\}^5 \).

If \( \{x, y\} \) is linearly independent, an immediate consequence of Lemma 1 is that \( \dim(\ker(A - I)) = \dim(\{x, y\}^5) = n - 2 \). Thus, there are \( n - 2 \) Jordan blocks corresponding to 1 in the Jordan canonical form of \( A \).

For completeness, we include the following lemma, which is used several times in the paper and can be readily verified. If \( A = [a_{ij}] \in M_n \), the **trace of \( A \)** is defined to be \( \text{tr}A = \Sigma_{j=1}^n a_{jj} \).

**Lemma 2.** Let \( A, B \in M_2 \) be given such that neither is a scalar matrix. Then \( A \) and \( B \) are similar if and only if \( \text{tr}A = \text{tr}B \) and \( \det A = \det B \).

Let \( \{x, y\} \) be a linearly independent subset of \( \mathbb{C}^n \). We consider each of the three possibilities (i) \( H_x, H_y \in K_S \) (ii) \( H_x, H_y \in L_S \) or (iii) \( H_x \in K_S \) and \( H_y \in L_S \) and determine the Jordan canonical form of the product \( H_x H_y \).

\( H_x, H_y \in K_S \)

Let \( \{x, y\} \) be a linearly independent subset of \( \mathbb{C}^n \) such that \( H_x, H_y \in K_S \), i.e., \( H_x = I + ir_x xx^*S \) and \( H_y = I + ir_y yy^*S \), where \( x^*Sx = y^*Sy = 0 \), and \( r_x, r_y \) are nonzero real numbers. If \( A = H_x H_y \), then

\[
A = I + ir_x xx^*S + ir_y yy^*S - r_x r_y (x^*Sy)xy^*S.
\]

Either \( x^*Sy = 0 \) or \( x^*Sy \neq 0 \).
Case 1: If \( x^*S y = 0 \), then \( A = I + i \tilde{r} x x^* S + i \tilde{r} y y^* S \). Note that \( \{x, y\}^S = \{x\}^S \cap \{y\}^S \), which is of dimension \( n - 2 \). If \( n > 2 \), then there exists \( z \in \{y\}^S \) but \( z \notin \{x\}^S \). Hence, \( (A - I)z = i \tilde{r} (x^*Sz)x \neq 0 \). Since \( x^*Sx = y^*Sy = x^*Sy = 0 \), we have \( (A - I)^2 \neq 0 \). Since \( A - I \neq 0 \), the minimal polynomial of \( A \) is \( x - 1 \) and so the largest Jordan block corresponding to 1 is of size 2. The number of Jordan blocks corresponding to 1 of size 1 is \( \text{rank}(A - I)^0 - 2 \text{rank}(A - I) + \text{rank}(A - I)^2 = n - 2(2) + 0 = n - 4 \). Since 1 is the only eigenvalue of \( A \) and there are \( n - 2 \) Jordan blocks corresponding to 1, \( A \) is similar to \( I_{n-2} \oplus J_2 (1) \oplus J_2 (1) \). If \( n = 2 \), then \( x^*S y \neq 0 \), otherwise \( x^*Sx = y^*Sy = x^*Sy = 0 \) and \( \{x, y\} \) linearly independent imply \( C^2 = \{x, y\}^S \) is of dimension \( n - 2 = 0 \), which is a contradiction.

Case 2: Suppose \( x^*S y \neq 0 \). We find any remaining eigenvalues of \( A \). The images of \( x \) and \( y \) under \( A \) are

\[
Ax = x + i \tilde{r}_x (x^*Sx)x + i \tilde{r}_y (y^*Sx)y - r_x r_y (x^*Sy)(y^*Sx)x = (1 - r_x r_y \|x^*Sy\|^2) x + i \tilde{r}_y (y^*Sx)y
\]

and

\[
Ay = y + i \tilde{r}_x (x^*Sy)x + i \tilde{r}_y (y^*Sy)y - r_x r_y (x^*Sy)(y^*Sy)x = y + i \tilde{r}_y (x^*Sy)x.
\]

Hence \( \text{span}\{x, y\} \) is invariant under \( A \). Consider the restriction of \( A \) to \( \text{span}\{x, y\} \) and its matrix representation

\[
M = \begin{bmatrix}
1 - r_x r_y |x^*Sy|^2 & i \tilde{r}_y (x^*Sy) \\
ir_y (y^*Sy) & 1
\end{bmatrix}
\]

with respect to the ordered basis \( \{x, y\} \). Since \( x^*Sx = y^*Sy = 0 \) and \( x^*Sy \neq 0 \), we have \( C^2 = \text{span}\{x, y\} \oplus \{x, y\}^S \). Thus \( A \) is similar to \( M \oplus I_{n-2} \) and 1 is not an eigenvalue of \( M \). Note that \( \det(M) = 1 \) and \( \text{tr}(M) = 2 - r_x r_y \|x^*Sy\|^2 \in \mathbb{R} \). Since \( A \in U_2 \) has determinant 1 and \( M \) is not a scalar matrix, we see that \( M \) is similar to one of the following: \( \text{diag}(e^{i\theta}, e^{-i\theta}) \), where \( \theta \in \mathbb{R} \) such that \( e^{i\theta} \neq \pm 1 \); \( J_2 (-1) \); or \( \text{diag}(\lambda, \lambda^{-1}) \), where \( \lambda \in \mathbb{R} \) and \( |\lambda| > 1 \). We determine if the preceding three possibilities for the Jordan canonical form of \( M \) occur.

Let \( \theta \in \mathbb{R} \) such that \( e^{i\theta} \neq \pm 1 \). If we choose \( r_x, r_y \in \mathbb{R} \) such that \( r_x r_y = \frac{2(1 - \cos \theta)}{|x^*Sy|^2} \neq 0 \), then \( \det(M) = 1 = \det(\text{diag}(e^{i\theta}, e^{-i\theta})) \) and \( \text{tr}(M) = 2 \cos \theta = \text{tr}(\text{diag}(e^{i\theta}, e^{-i\theta})) \). By Lemma 2, \( M \) is similar to \( \text{diag}(e^{i\theta}, e^{-i\theta}) \).
If we choose \( r_x, r_y \in \mathbb{R} \) such that \( r_x r_y = \frac{4}{|x^*Sy|^2} \), then \( \text{tr}(M) = -2 = \text{tr}(J_2 (-1)) \) and \( \det(M) = 1 = \det (J_2 (-1)) \). By Lemma 2, \( M \) is similar to \( J_2 (-1) \).

Let \( \lambda \in \mathbb{R} \) such that \(|\lambda| > 1\). If we choose \( r_x, r_y \in \mathbb{R} \) such that \( r_x r_y = \frac{(2 - \lambda - \lambda^{-1})}{|x^*Sy|^2} \neq 0\), then we have \( \det(M) = 1 = \det (\text{diag} (\lambda, \lambda^{-1})) \) and \( \text{tr}(M) = -2 = \text{tr}(\text{diag} (\lambda, \lambda^{-1})) \). Since \( \lambda \neq \lambda^{-1} \), we have that \( M \) is similar to \( \text{diag} (\lambda, \lambda^{-1}) \).

**Theorem 3.** Let \( S \in \text{GL}_n \) be indefinite Hermitian and \( x, y \in \mathbb{C}^n \) be given. If \( \{x, y\} \) is linearly independent and \( H_x, H_y \in \mathcal{K}_S \), then the product \( H_x H_y \) is similar to one of the following:

a. \( I_{n-4} \oplus J_2 (1) \oplus J_2 (1) \)

b. \( I_{n-2} \oplus J_2 (-1) \)

c. \( I_{n-2} \oplus \text{diag}(e^{i\theta}, e^{-i\theta}), \text{where } \theta \in \mathbb{R} \text{ such that } e^{i\theta} \neq \pm 1 \)

d. \( I_{n-2} \oplus \text{diag}(\lambda, \lambda^{-1}), \text{where } |\lambda| > 1 \text{ and } \lambda \in \mathbb{R} \).

\( H_x, H_y \in \mathcal{L}_S \)

We now consider the product of two elements of \( \mathcal{L}_S \). Let \( x, y \in \mathbb{C}^n \) such that \( \{x, y\} \) is linearly independent and \( H_x, H_y \in \mathcal{L}_S \), that is, \( H_x = I + \frac{e^{i\alpha} - 1}{x^*Sx} xx^*S \) and \( H_y = I + \frac{e^{i\beta} - 1}{y^*Sy} yy^*S \), where \( x^*Sx \) and \( y^*Sy \) are nonzero, and \( \alpha, \beta \in \mathbb{R} \) such that \( e^{i\alpha} \neq 1 \) and \( e^{i\beta} \neq 1 \). Since \( H_y = H_y \) for all nonzero \( a \in \mathbb{C} \), we can assume that \( x^*Sx, y^*Sy \in \{1, -1\} \). If \( A = H_x H_y \) then

\[
A = I + \frac{e^{i\alpha} - 1}{x^*Sx} xx^*S + \frac{e^{i\beta} - 1}{y^*Sy} yy^*S + \frac{e^{i\alpha} - 1}{x^*Sx} \frac{e^{i\beta} - 1}{y^*Sy} (x^*Sy)xy^*S.
\]

**Case 1:** If \( x^*Sy = 0 \), then \( A = I + \frac{e^{i\alpha} - 1}{x^*Sx} xx^*S + \frac{e^{i\beta} - 1}{y^*Sy} yy^*S \). Observe that \( Ax = x + (e^{i\alpha} - 1)x = e^{i\alpha}x \). Hence, \( x \) is an eigenvector of \( A \) corresponding to \( e^{i\alpha} \). Similarly, \( y \) is an eigenvector of \( A \) corresponding to \( e^{i\beta} \). Since \( x^*Sx \) and \( y^*Sy \) are nonzero and \( x^*Sy = 0 \), we have \( \mathbb{C}^n = \text{span} \{x, y\} \oplus \{x, y\}^5 \). Hence \( A \) is similar to \( I_{n-2} \oplus \text{diag}(e^{i\alpha}, e^{i\beta}) \).
Case 2: Suppose $x^*Sy \neq 0$. We find any remaining eigenvalues of $A$. The images of $x$ and $y$ under $A$ are

$$A x = x + (e^{i\alpha} - 1)x + \frac{e^{i\beta} - 1}{y^*Sy} (y^*Sx)y + \frac{e^{i\alpha} - 1}{x^*Sx} \frac{e^{i\beta} - 1}{y^*Sy} (x^*Sy)(y^*Sx)x$$

$$= \left(e^{i\alpha} + \frac{e^{i\alpha} - 1}{x^*Sx} \frac{e^{i\beta} - 1}{y^*Sy} |x^*Sy|^2\right)x + \left(\frac{e^{i\beta} - 1}{y^*Sy} (y^*Sx)y\right)$$

and

$$A y = y + \frac{e^{i\alpha} - 1}{x^*Sx} (x^*Sy)x + \frac{e^{i\beta} - 1}{y^*Sy} (y^*Sx)y + \frac{e^{i\alpha} - 1}{x^*Sx} \frac{e^{i\beta} - 1}{y^*Sy} (x^*Sy)(y^*Sx)x$$

$$= e^{i\beta} y + \left(e^{i\beta} x^*Sy - \frac{e^{i\alpha} - 1}{x^*Sx}\right)x.$$

Hence $\text{span}\{x, y\}$ is invariant under $A$. Consider the restriction of $A$ to $\text{span}\{x, y\}$ and its matrix representation

$$L = \begin{bmatrix}
  e^{i\alpha} + \frac{e^{i\alpha} - 1}{x^*Sx} \frac{e^{i\beta} - 1}{y^*Sy} |x^*Sy|^2 & e^{i\beta} x^*Sy - \frac{e^{i\alpha} - 1}{x^*Sx} \\
  \frac{e^{i\beta} - 1}{y^*Sy} (y^*Sx) & e^{i\beta}
\end{bmatrix}$$

with respect to the ordered basis $\{x, y\}$.

Note that $a x + b y \in \{x, y\}^5$ for some $a, b \in \mathbb{C}$ if and only if $x^*S(a x + b y) = 0$ and $y^*S(a x + b y) = 0$, that is

$$\begin{bmatrix}
  x^*Sx & x^*Sy \\
  y^*Sx & y^*Sy
\end{bmatrix}\begin{bmatrix}
  a \\
  b
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.$$  Since $x^*Sx, y^*Sy \in \{1, -1\}$, we have $\{x, y\}^5 \cap \text{span}\{x, y\} = \{0\}$ if and only if $x^*Sx$ and $y^*Sy$ have opposite signs or $|x^*Sy| \neq 1$.

If $x^*Sx = y^*Sy \in \{\pm 1\}$ and $|x^*Sy| = 1$, then $x, y \notin \{x, y\}^5$ and

$\{x, y\}^5 \cap \text{span}\{x, y\} = \text{span}\{(x^*Sy)x - (x^*Sx)y\}$. 
Hence \( \text{span}\{x\} \oplus \{x, y\}^s \) is of dimension \( n-1 \) and contains \( \text{span}\{x, y\} \). Now \( Ax \) can be rewritten as \( Ax = e^{(a+i\beta)I}x - (e^{i\beta} - 1)(y^*Sx)[(x^*Sy)x - (x^*Sx)y] \). This implies that \( \text{span}\{x\} \oplus \{x, y\}^s \) is invariant under \( A \). Since \( \det A = e^{i(a+\beta)} \) and \( \text{rank}(A-I) = 2 \), we have that \( A \) is similar to \( l_{n-2} \oplus J_x(1) \oplus [e^{(a+i\beta)}], \) if \( e^{(a+i\beta)} \neq 1 \); or \( l_{n-3} \oplus J_y(1) \), if \( e^{(a+i\beta)} = 1 \).

If \( \mathbb{C}^n = \text{span}\{x, y\} \oplus \{x, y\}^s \), then \( A \) is similar to \( l_{n-2} \oplus L \) and \( L \) is not an eigenvalue of \( L \). Observe that \( \det L = e^{i(a+\beta)} \) and \( \text{tr} L = e^{ia} + e^{i\beta} + \frac{e^{ia} - 1}{x^*Sx}, \frac{e^{i\beta} - 1}{y^*Sy} \) and \( \text{tr} L = |x^*Sy|^2 \).

Since \( A \in U_S \) and \( L \) is not a scalar matrix, then \( L \) is similar to one of the following:

\[ \text{diag}(e^{i\beta}, e^{i\phi}), \text{where } \theta, \phi \in \mathbb{R} \text{ such that } e^{i\beta}, e^{i\phi} \text{ are distinct and both are not equal to } 1; \]

\[ J_x(\lambda), \text{where } |\lambda| = 1 \text{ but } \lambda \neq 1; \text{ or } \text{diag}(\lambda, \overline{\lambda} - 1), \text{where } |\lambda| > 1. \]

It suffices to determine whether the last two possibilities for the Jordan canonical form of \( L \) occur. But first we need to determine the possible nonzero values of \( x^*Sy, y^*Sy \in \{1, -1\} \) and \( \{x, y\} \) is linearly independent. Let \( e_i \in \mathbb{C}^n \) denote the column with \( i \)th entry 1 and 0 elsewhere. Suppose \( c \in \mathbb{C} \) is nonzero and \( S = P^*(l_{k} \oplus -l_{n-k})P \), for some nonsingular \( P \) and integer \( 0 < k < n \). If \( |c| > 1 \), we can take \( x, y \in \mathbb{C}^n \) such that \( Px = e_1 \) and \( Py = ce_1 + \sqrt{|c|^2 - |1e_{k+1}|}, \) so that \( x^*Sx = 1, y^*Sy = |c|^2 - (|c|^2 - 1) = 1, \) and \( x^*Sy = c. \) Thus, if \( |c| > 1 \), there exists a linearly independent set \( \{x, y\} \) such that \( x^*Sx = y^*Sy \) and \( x^*Sy = c. \) If \( c \in \mathbb{C} \) is nonzero and we take \( x, y \in \mathbb{C}^n \) such that \( Px = e_1 \) and \( Py = ce_1 + \sqrt{|c|^2 - |1e_{k+1}|}, \) then \( x^*Sx = 1, y^*Sy = |c|^2 - (|c|^2 + 1) = -1 \) and \( x^*Sy = c. \) Hence every nonzero \( c \in \mathbb{C} \) can be realized as \( x^*Sy \) by a linearly independent set \( \{x, y\} \) such that \( x^*Sx = -y^*Sy \), when \( S \) is \( * \)-congruent to \( l_k \oplus -l_{n-k}. \)

Let \( \alpha = \beta \in \mathbb{R} \) such that \( \alpha \neq k\pi, \) for all \( k \in \mathbb{Z}. \) If \( a = \text{Re}(e^{ia}), \) then \( \frac{-4e^{ia}}{(e^{ia} - 1)^2} = \frac{2}{1-a} > 1. \)

If we take \( x, y \in \mathbb{C}^n \) such that \( x^*Sx = 1 = y^*Sy \) and \( |x^*Sy|^2 = \frac{-4e^{ia}}{(e^{ia} - 1)^2}, \) then \( \text{tr} L = 2e^{ia} + (e^{ia} - 1)2|x^*Sy|^2 = -2e^{ia} \) and \( \text{det} L = e^{2ia}. \) Since \( L \) is not a scalar matrix, it follows from Lemma 2 that \( L \) is similar to \( J_y(-e^{ia}), \) where \( e^{ia} \neq \pm 1. \)

If we take \( e^{ia} = e^{-i\beta} = i, \) and \( x, y \in \mathbb{C}^n \) such that \( x^*Sx = 1 = -y^*Sy \) and \( |x^*Sy| = 1, \) then \( \text{tr} L = -2 \) and \( \text{det} L = 1. \) Since \( L \) is not a scalar matrix, \( L \) is similar to \( J_y(-1). \)

Let \( \lambda = re^{ia}, \) where \( r > 1 \) and \( \theta \neq 2k\pi \) for all \( k \in \mathbb{Z}. \) Then \( \frac{e^{ia}(r - 1)^2}{(e^{ia} - 1)^2r} \) is positive.
If we take $\alpha = \beta = 0$, and $x, y \in \mathbb{C}^n$ such that $x^*Sx = 1 = -y^*Sy$ and $|x^*Sy|^2 = -\frac{e^{\alpha}(r-1)^2}{(e^{\alpha}-1)2r}$, then $\text{tr} \ L = 2e^{\alpha} - (e^{\alpha}-1)^2 |x^*Sy|^2 = e^{\alpha} (r + r^{-1}) = \lambda + \overline{\lambda}^{-1}$ and $\det L = e^{2\alpha} = \lambda \overline{\lambda}^{-1}$. Hence $L$ is similar to diag$(\lambda, \overline{\lambda}^{-1})$.

Let $\lambda = r$, where $r > 1$. Let $\beta = -\alpha$ and $\alpha \in \mathbb{R}$ such that Re$(e^{i\alpha}) = r$. Since $\frac{(r-1)^2}{r} > 0$, we have $\frac{r - r^{-1}}{2(1-r^{-1})} > 1$. If we take $x, y \in \mathbb{C}^n$ such that $x^*Sx = 1 = y^*Sy$ and $|x^*Sy|^2 = \frac{r - r^{-1}}{2(1-r^{-1})}$, then $\text{tr} \ L = 2r^{-1} + 2(1-r^{-1}) |x^*Sy|^2 = r + r^{-1}$ and $\det L = 1$. Hence $L$ is similar to diag$(r, r^{-1})$.

**Theorem 4.** Let $S \in GL_n$ be indefinite Hermitian and $x, y \in \mathbb{C}^n$ be given. If $\{x, y\}$ is linearly independent such that $H_x, H_y \in L_S$, then the product $H_x H_y$ is similar to one of the following:

a. $I_{n-3} \oplus \text{diag}(e^{i\theta}, e^{i\phi})$, where $\theta, \phi \in \mathbb{R}$ such that $e^{i\theta}, e^{i\phi} \neq 1$

b. $I_{n-3} \oplus J_2(1)$ $\oplus [e^{i\theta}]$, where $\theta \in \mathbb{R}$ and $e^{i\theta} \neq 1$

c. $I_{n-3} \oplus J_3(1)$

d. $I_{n-2} \oplus J_2(\lambda)$, where $|\lambda| = 1$ and $\lambda \neq 1$

e. $I_{n-2} \oplus \text{diag}(\lambda, \overline{\lambda}^{-1})$, where $|\lambda| > 1$.

$H_x \in K_S$ and $H_y \in L_S$

Lastly, we consider the product of an element of $K_S$ and of $L_S$. If $x, y \in \mathbb{C}^n$ are nonzero such that $H_x \in K_S$ and $H_y \in L_S$, then $H_x = I + irxx^*S$, where $r \in \mathbb{R} \setminus \{0\}$, and $x^*Sx = 0$, and $H_y = I + \frac{e^{i\alpha} - 1}{y^*Sy} yy^*S$, where $e^{i\alpha} \neq 1$. Note that $\{x, y\}$ is linearly independent since $x^*Sx = 0 \neq y^*Sy$. If $A = H_x H_y$, then

$$A = I + irxx^*S + \frac{e^{i\alpha} - 1}{y^*Sy} yy^*S + ir \frac{e^{i\alpha} - 1}{y^*Sy} (x^*Sy) xy^*S.$$
Observe that \( \text{rank}(A - I) = 2 \) and \( \text{rank}(A - I)^2 = \text{rank}(A - I)^3 = 1 \), which imply that 2 is the size of the largest Jordan block corresponding to 1, and the number of Jordan blocks of size 2 corresponding to 1 is \( \text{rank}(A - I) - 2\text{rank}(A - I)^2 + \text{rank}(A - I)^3 = 2 - 2(1) + 1 = 1 \). Since there are \( n - 2 \) Jordan blocks corresponding to 1 and \( \det A = e^{i\alpha} \), we have that \( A \) is similar to \( I_{n-3} \oplus J_2(I) \oplus [e^{i\alpha}] \).

**Case 2:** Suppose \( x^*Sy \neq 0 \). The images of \( x \) and \( y \) under \( A \) are

\[
Ax = x + \frac{e^{i\alpha} - 1}{y^*Sy}(y^*Sx)y + ir\frac{e^{i\alpha} - 1}{y^*Sy}|x^*Sy|^2x
\]

\[
= \left(1 + ir\frac{e^{i\alpha} - 1}{y^*Sy}|x^*Sy|^2\right)x + \frac{e^{i\alpha} - 1}{y^*Sy}(y^*Sx)y
\]

and

\[
Ay = y + ir(x^*Sy)x + (e^{i\alpha} - 1)y + ir(e^{i\alpha} - 1)(x^*Sy)x = ire^{i\alpha}(x^*Sy)x + e^{i\alpha}y.
\]

Hence \( \text{span}\{x, y\} \) is invariant under \( A \). Consider the restriction of \( A \) to \( \text{span}\{x, y\} \) and its matrix representation

\[
K = \begin{bmatrix}
1 + ir\frac{e^{i\alpha} - 1}{y^*Sy}|x^*Sy|^2 & ire^{i\alpha}(x^*Sy) \\
\frac{e^{i\alpha} - 1}{y^*Sy}(y^*Sx) & e^{i\alpha}
\end{bmatrix}
\]

with respect to the ordered basis \( \{x, y\} \). Since \( \mathbb{C}^n = \text{span}\{x, y\} \oplus \{x, y\}^\perp \), \( A \) is similar to \( I_{n-2} \oplus K \) and 1 is not an eigenvalue of \( K \). Note that \( \det K = e^{i\alpha} \neq 1 \) and \( \text{tr} K = e^{i\alpha} + 1 + ir\frac{e^{i\alpha} - 1}{y^*Sy}|x^*Sy|^2 \). Since \( A \) is \( S \)-unitary, \( K \) is similar to one of the following:

- \( \text{diag}(e^{i\theta}, e^{i\phi}) \), where \( \theta, \phi \in \mathbb{R} \) such that \( e^{i\theta}, e^{i\phi} \) are distinct with both not equal to 1, and \( e^{i(\theta + \phi)} = e^{i\alpha} \); or
- \( \text{diag}(\lambda, \lambda^{-1}) \), where \( |\lambda| > 1 \) and \( \lambda \neq \pm 1 \).

We now determine whether the three possibilities for the Jordan canonical form of \( K \) occur.
Let $\theta, \phi \in \mathbb{R}$ such that $e^{i\theta}, e^{i\phi},$ and $e^{i(\theta + \phi)}$ are not equal to 1, and $e^{i\theta} \neq e^{i\phi}$. If $\alpha = \theta + \phi$, choose $r \in \mathbb{R}$ such that $r(e^{i\alpha} - 1) = \frac{(y^*Sy)(1-e^{i\alpha})(e^{i\phi}-1)}{i|x^*Sy|^2}$. This has a solution since $\frac{(1-e^{i\alpha})(e^{i\phi}-1)}{e^{i\alpha}-1} = -1 + \frac{e^{i\phi}+e^{i\alpha}-2}{e^{i\alpha}-1}$ is nonzero and the real part of $\frac{e^{i\phi}+e^{i\alpha}-2}{e^{i\alpha}-1}$ is 1. Then $\det K = e^{i(\theta + \phi)} = \det(\text{diag}(e^{i\theta}, e^{i\phi}))$ and $\text{tr} K = e^{i\theta} + e^{i\phi} = \text{tr}(\text{diag}(e^{i\theta}, e^{i\phi}))$. Thus $K$ is similar to $\text{diag}(e^{i\theta}, e^{i\phi})$.

Let $\lambda = te^{i\gamma}$, where $t, \gamma \in \mathbb{R}$ such that $t > 1$ and $e^{i\gamma} \neq 1$. Choose $\alpha = 2\gamma$ and $r \in \mathbb{R}$ such that $r(e^{i\alpha} - 1) = \frac{(y^*Sy)(1-te^{i\gamma})(t^{-1}e^{i\gamma}-1)}{i|x^*Sy|^2}$. This has a solution since $\frac{(1-te^{i\gamma})(t^{-1}e^{i\gamma}-1)}{e^{i\alpha}-1} = -1 + \frac{(t+t^{-1})e^{i\gamma}-2}{te^{i\gamma}-1}$ is nonzero and the real part of $\frac{(t+t^{-1})e^{i\gamma}-2}{te^{i\gamma}-1}$ is 1. Then $\det K = \lambda \overline{\lambda}^{-1} = \det(\text{diag}(\lambda, \overline{\lambda}^{-1}))$ and $\text{tr} K = (t + t^{-1}) e^{i\gamma} = \text{tr} (\text{diag}(\lambda, \overline{\lambda}^{-1}))$. By Lemma 2, $K$ is similar to $\text{diag}(\lambda, \overline{\lambda}^{-1})$.

Let $\lambda = e^{i\beta}$, where $\beta \in \mathbb{R}$ and $\lambda \neq \pm 1$. Choose $\alpha = 2\beta$ and $r \in \mathbb{R}$ such that $r = \frac{(1-\lambda)y^*Sy}{i(\lambda + 1)|x^*Sy|^2}$. This has a solution since $\frac{1-\lambda}{\lambda + 1} = -1 + \frac{2}{\lambda + 1}$ and the real part of $\frac{2}{\lambda + 1}$ is 1. Then $\det K = \lambda^2 = \det J_2(\lambda)$ and $\text{tr} K = 2\lambda = \text{tr} J_2(\lambda)$. Since $K$ is not a scalar matrix, $K$ is similar to $J_2(\lambda)$.

**Theorem 5.** Let $S \in GL_n$ be indefinite Hermitian and $x, y \in \mathbb{C}^n$ be given. If $\{x, y\}$ is linearly independent such that $H_x \in K_z$ and $H_y \in L_z$, then the product $H_x H_y$ is similar to one of the following:

a. $I_{n-3} \oplus J_2(1) \oplus [e^{i\alpha}]$, for some $\alpha \in \mathbb{R}$ such that $e^{i\alpha} \neq 1$

b. $I_{n-2} \oplus \text{diag}(e^{i\theta}, e^{i\phi})$, where $\theta, \phi \in \mathbb{R}$ such that $e^{i\theta}, e^{i\phi}, e^{i(\theta + \phi)}$ are all not equal to 1, and $e^{i\theta} \neq e^{i\phi}$

c. $I_{n-2} \oplus \text{diag}(\lambda, \overline{\lambda}^{-1})$, where $|\lambda| > 1$ and $\lambda \notin \mathbb{R}$

d. $I_{n-2} \oplus J_2(\lambda)$, where $|\lambda| = 1$ but $\lambda \neq \pm 1$. 


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