The Jordan Canonical Form of a Product of Elementary *S*-unitary Matrices

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ABSTRACT

Let S be an n-by-n, nonsingular, and Hermitian matrix. A square complex matrix Q is said to be S-unitary if $Q^*SQ = S$. An S-unitary matrix Q is said to be elementary if rank$(Q - I) = 1$. It is known what form every elementary S-unitary can take, and that every S-unitary can be written as a product of elementary S-unitaries. In this paper, we determine the Jordan canonical form of a product of two elementary S-unitaries.

Keywords: elementary S-unitary matrix, Hermitian matrix, Jordan canonical form

INTRODUCTION

Let M_n be the set of all n-by-n matrices with entries in the complex field $\mathbb{C}$ and let GL_n be the set of all nonsingular matrices in M_n. Let $S \in GL_n$ be Hermitian. A $Q \in M_n$ is said to be S-unitary if $Q^*SQ = S$, where Q^* is the conjugate transpose of Q (Gohberg et al. 2005). If $S = I$, then the set of S-unitary matrices in GL_n coincides with the set of unitary matrices. Let U_S be the set of all S-unitary matrices. Observe that U_S is nonempty since $I \in U_S$. If $Q \in U_S$, then Q_{-1} is similar to Q^*, $|\det Q| = 1$, and $\alpha Q \in U_S$ for all $\alpha \in \mathbb{C}$ with modulus 1. Moreover, U_S is a group under multiplication and consists of all matrices in M_n that preserve the scalar product $[\mathbf{u},\mathbf{v}]_S = \mathbf{u}^*S\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$.

An $H \in U_S$ is called elementary if rank$(H - I) = 1$. Let H_S be the set of all elementary S-unitary matrices. When S is Hermitian, $H_S = K_S \cup L_S$, where

$$K_S = \{K_{\mathbf{x},r} = I + ir\mathbf{x}\mathbf{x}^*S : \mathbf{x} \in \mathbb{C}^n\backslash\{\mathbf{0}\}, \mathbf{x}^*S\mathbf{x} = 0, \text{and } r \in \mathbb{R}\backslash\{0\}\}$$

and

$$LS = \{L_{\mathbf{x},\varphi} = I + \frac{(e^{i\varphi} - 1)}{\mathbf{x}^*S\mathbf{x}}\mathbf{x}\mathbf{x}^*S: \mathbf{x} \in \mathbb{C}^n, \mathbf{x}^*S\mathbf{x} \neq 0, \varphi \in \mathbb{R}, \text{and } e^{i\varphi} \neq 1\}$$

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(Catbagan 2015). If $\mathbf{v} \in \mathbb{C}^n$ such that $\mathbf{v}^*S\mathbf{v} \neq 0$, the Λ_S-Householder matrix $S\mathbf{v} = I - \dfrac{2}{\mathbf{v}^*S\mathbf{v}}\mathbf{v}\mathbf{v}^*S$ generalizes the Householder matrix of $\mathbf{v}$, which is equal to $L_{\mathbf{v},\pi}$ for $S = I$ (Merino et al. 2011; Horn and Johnson 2013). If $K_{\mathbf{x},r} \in K_S$, then $K_{\mathbf{x},r}^{-1} = K_{\mathbf{x},-r}$ and $K_{\mathbf{x},r}$ is similar to $I_{n-2} \oplus J_2(1)$. If $L_{\mathbf{x},\varphi} \in L_S$, then $L_{\mathbf{x},\varphi}-1 = L_{\mathbf{x},-\varphi}$ and $L_{\mathbf{x},\varphi}$ is similar to $I_{n-1} \oplus [ei\varphi]$. Hence $H \in H_S$ if and only if $H^{-1} \in H_S$. Thus, I is a product of two elements of H_S. Moreover, if $A \in U_S$, then A can be written as a product of elements of H_S (Catbagan 2015). Thus, the elements of H_S generate the group U_S. Since there are two types of elements of H_S, there are three types of products of two elements of H_S up to similarity. We wish to determine which Jordan canonical forms arise for each possibility, since the Jordan canonical form of a matrix reveals a lot of information such as its rank, nullity, eigenvalues, and their algebraic and geometric multiplicities. Analogous results for symplectic matrices and *J*-Householder matrices can be found in de la Rosa et al. (2012).

PRELIMINARIES

If $S = P^*P$ for some $P \in GL_n$, then $\mathbf{x}^*S\mathbf{x} > 0$, when $0 \neq \mathbf{x} \in \mathbb{C}^n$; and $Q \in U_S$ if and only if $PQP^{-1} \in U_I$. Hence when S is positive definite, $H_S = L_S$, and every *S*-unitary is similar to a unitary matrix. Since $U_S = U_{-S}$, from now on we only consider S that is *-congruent to $I_k \oplus -I_{n-k}$ for $0 < k < n$, that is $S = P^*(I_k \oplus -I_{n-k})P$, for some $P \in GL_n$.

Let n be a positive integer such that $n \geq 2$, and $T \subseteq \mathbb{C}^n$ be nonempty. Let T^S be the *S*-perpendicular subspace of T defined by

$$T^S = \{\, \mathbf{z} \in \mathbb{C}^n \mid \mathbf{x}^*S\mathbf{z} = 0, \text{ for all } \mathbf{x} \in T\}.$$

Then $\dim T^S = n - \dim(\operatorname{span} T)$, since $T^S = (S(\operatorname{span} T))^{\perp}$, which is the orthogonal complement of $S(\operatorname{span} T)$ with respect to the usual inner product on $\mathbb{C}^n$, and $\mathbb{C}^n = W \oplus W^{\perp}$ for any subspace W of $\mathbb{C}^n$. Let $H_{\mathbf{x}}, H_{\mathbf{y}} \in H_S$ and $A = H_{\mathbf{x}} H_{\mathbf{y}}$. Then $H_{\mathbf{x}} = I + \alpha\mathbf{x}\mathbf{x}^*S$ and $H_{\mathbf{y}} = I + \beta\mathbf{y}\mathbf{y}^*S$, for some nonzero $\alpha, \beta \in \mathbb{C}$. If $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent, then $\mathbf{y} = \delta\mathbf{x}$, for some $\delta \in \mathbb{C}$. This implies

$$A = I + \alpha\mathbf{x}\mathbf{x}^*S + \beta\mathbf{y}\mathbf{y}^*S + \alpha\beta\mathbf{x}\mathbf{x}^*S\mathbf{y}\mathbf{y}^*S = I + (\alpha + \beta|\delta|^2 + \alpha\beta|\delta|^2\, \mathbf{x}^*S\mathbf{x})\mathbf{x}\mathbf{x}^*S.$$

Hence $A = I + \mu\mathbf{x}\mathbf{x}^*S$, where $\mu = \alpha + \beta|\delta|^2 + \alpha\beta|\delta|^2\, \mathbf{x}^*S\mathbf{x} \in \mathbb{C}$. If $\mu = 0$, then $A = I$, which implies $H_{\mathbf{x}} = H_{\mathbf{y}}^{-1}$. If $\mu \neq 0$, then $\operatorname{rank}(A - I) = 1$, and since $A \in U_S$, we have $A \in H_S$.

Suppose $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent. Let $\mathbf{z} \in \mathbb{C}^n$ be given. Suppose $\mathbf{z} \in \ker(A - I)$, that is, $A\mathbf{z} = \mathbf{z}$. Then

$$\mathbf{0} = (A - I)\mathbf{z} = (\alpha\mathbf{x}^*S\mathbf{z})\mathbf{x} + (\beta\mathbf{y}^*S\mathbf{z})\mathbf{y} + \alpha\beta(\mathbf{x}^*S\mathbf{y})(\mathbf{y}^*S\mathbf{z})\mathbf{x}.$$

Since $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, and α, β are nonzero, we have $\mathbf{y}^*S\mathbf{z} = 0$ and it follows that $\mathbf{x}^*S\mathbf{z} = 0$. Thus, $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}^S$. Conversely, suppose $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}^S$. Then $\mathbf{x}^*S\mathbf{z} = \mathbf{y}^*S\mathbf{z} = 0$ and so

$$(A - I)\mathbf{z} = \alpha(\mathbf{x}^*S\mathbf{z})\mathbf{x} + \beta(\mathbf{y}^*S\mathbf{z})\mathbf{y} + \alpha\beta(\mathbf{x}^*S\mathbf{y})(\mathbf{y}^*S\mathbf{z})x = \mathbf{0},$$

that is, $\mathbf{z} \in \ker(A - I)$. Therefore $\ker(A - I) = \{\mathbf{x}, \mathbf{y}\}^S$.

Lemma 1. Let $S \in GL_n$ *be Hermitian and let* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *be nonzero. Suppose* $H_x, H_y \in H_S$ *and* $A = H_x H_y$ *. If* $\{\mathbf{x}, \mathbf{y}\}$ *is linearly dependent, then* $A = I$ *or* $A \in H_S$*. If* $\{\mathbf{x}, \mathbf{y}\}$ *is linearly independent, then* $\ker(A - I) = \{\mathbf{x}, \mathbf{y}\}^S$.

If $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, an immediate consequence of Lemma 1 is that $\dim(\ker(A - I)) = \dim(\{\mathbf{x}, \mathbf{y}\}^S) = n - 2$. Thus, there are $n - 2$ Jordan blocks corresponding to 1 in the Jordan canonical form of A.

For completeness, we include the following lemma, which is used several times in the paper and can be readily verified. If $A = [a_{ij}] \in M_n$, the *trace of* A is defined to be $\mathrm{tr}A = \Sigma_{j=1}^{n} a_{jj}$.

Lemma 2. Let $A, B \in M_2$ be given such that neither is a scalar matrix. Then A and B are similar if and only if $\mathrm{tr}A = \mathrm{tr}B$ and $\det A = \det B$.

Let $\{\mathbf{x}, \mathbf{y}\}$ be a linearly independent subset of $\mathbb{C}^n$. We consider each of the three possibilities (i) $H_x, H_y \in K_S$, (ii) $H_x, H_y \in L_S$, or (iii) $H_x \in K_S$ and $H_y \in L_S$, and determine the Jordan canonical form of the product H_xH_y.

$\boldsymbol{H_x, H_y \in K_S}$

Let $\{\mathbf{x}, \mathbf{y}\}$ be a linearly independent subset of $\mathbb{C}^n$ such that $H_x, H_y \in K_S$, i.e., $H_x = I + ir_x\mathbf{x}\mathbf{x}^*S$ and $H_y = I + ir_y\mathbf{y}\mathbf{y}^*S$, where $\mathbf{x}^*S\mathbf{x} = \mathbf{y}^*S\mathbf{y} = 0$, and r_x, r_y are nonzero real numbers. If $A = H_x H_y$, then

$$A = I + ir_x\mathbf{x}\mathbf{x}^*S + ir_y\mathbf{y}\mathbf{y}^*S - r_xr_y(\mathbf{x}^*S\mathbf{y})\mathbf{x}\mathbf{y}^*S.$$

Either $\mathbf{x}^*S\mathbf{y} = 0$ or $\mathbf{x}^*S\mathbf{y} \neq 0$.

Case 1: If $\mathbf{x}^*S\mathbf{y} = 0$, then $A = I + ir_x\mathbf{x}\mathbf{x}^*S + ir_y\mathbf{y}\mathbf{y}^*S$. Note that $\{\mathbf{x}, \mathbf{y}\}^S = \{\mathbf{x}\}^S \cap \{\mathbf{y}\}^S$, which is of dimension $n - 2$. If $n > 2$, then there exists $\mathbf{z} \in \{\mathbf{y}\}^S$ but $\mathbf{z} \notin \{\mathbf{x}\}^S$. Hence, $(A - I)\mathbf{z} = ir_x(\mathbf{x}^*S\mathbf{z})\mathbf{x} \neq \mathbf{0}$. Since $\mathbf{x}^*S\mathbf{x} = \mathbf{y}^*S\mathbf{y} = \mathbf{x}^*S\mathbf{y} = 0$, we have $(A - I)^2 = 0$. Since $A - I \neq 0$, the minimal polynomial of A is $(x - 1)^2$ and so the largest Jordan block corresponding to 1 is of size 2. The number of Jordan blocks corresponding to 1 of size 1 is $\operatorname{rank}(A - I)^0 - 2\operatorname{rank}(A - I) + \operatorname{rank}(A - I)^2 = n - 2(2) + 0 = n - 4$. Since 1 is the only eigenvalue of A and there are $n - 2$ Jordan blocks corresponding to 1, A is similar to $I_{n-4} \oplus J_2(1) \oplus J_2(1)$. If $n = 2$, then $\mathbf{x}^*S\mathbf{y} \neq 0$, otherwise $\mathbf{x}^*S\mathbf{x} = \mathbf{y}^*S\mathbf{y} = \mathbf{x}^*S\mathbf{y} = 0$ and $\{\mathbf{x}, \mathbf{y}\}$ linearly independent imply $\mathbb{C}^2 = \{\mathbf{x}, \mathbf{y}\}^S$ is of dimension $n - 2 = 0$, which is a contradiction.

Case 2: Suppose $\mathbf{x}^*S\mathbf{y} \neq 0$. We find any remaining eigenvalues of A. The images of $\mathbf{x}$ and $\mathbf{y}$ under A are

$$A\mathbf{x} = \mathbf{x} + ir_x(\mathbf{x}^*S\mathbf{x})\mathbf{x} + ir_y(\mathbf{y}^*S\mathbf{x})\mathbf{y} - r_x r_y(\mathbf{x}^*S\mathbf{y})(\mathbf{y}^*S\mathbf{x})\mathbf{x} = (1 - r_x r_y|\mathbf{x}^*S\mathbf{y}|^2)\,\mathbf{x} + ir_y(\mathbf{y}^*S\mathbf{x})\mathbf{y}$$

and

$$A\mathbf{y} = \mathbf{y} + ir_x(\mathbf{x}^*S\mathbf{y})\mathbf{x} + ir_y(\mathbf{y}^*S\mathbf{y})\mathbf{y} - r_x r_y(\mathbf{x}^*S\mathbf{y})(\mathbf{y}^*S\mathbf{y})\mathbf{x} = \mathbf{y} + ir_y(\mathbf{x}^*S\mathbf{y})\mathbf{x}\,.$$

Hence $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$ is invariant under A. Consider the restriction of A to $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$ and its matrix representation

$$M = \begin{bmatrix} 1 - r_{\mathbf{x}} r_{\mathbf{y}} |\mathbf{x}^*S\mathbf{y}|^2 & ir_y(\mathbf{x}^*S\mathbf{y}) \\ ir_{\mathbf{y}}(\mathbf{y}^*S\mathbf{x}) & 1 \end{bmatrix}$$

with respect to the ordered basis $\{\mathbf{x}, \mathbf{y}\}$. Since $\mathbf{x}^*S\mathbf{x} = \mathbf{y}^*S\mathbf{y} = 0$ and $\mathbf{x}^*S\mathbf{y} \neq 0$, we have $\mathbb{C}^n = \operatorname{span}\{\mathbf{x}, \mathbf{y}\} \oplus \{\mathbf{x}, \mathbf{y}\}^S$. Thus A is similar to $M \oplus I_{n-2}$ and 1 is not an eigenvalue of M. Note that $\det(M) = 1$ and $\operatorname{tr}(M) = 2 - r_x r_y|\mathbf{x}^*S\mathbf{y}|^2 \in \mathbb{R}$. Since $A \in U_S$ has determinant 1 and M is not a scalar matrix, we see that M is similar to one of the following: $\operatorname{diag}(e^{i\theta}, e^{-i\theta})$, where $\theta \in \mathbb{R}$ such that $e^{i\theta} \neq \pm 1$; $J_2(-1)$; or $\operatorname{diag}(\lambda, \lambda^{-1})$, where $\lambda \in \mathbb{R}$ and $|\lambda| > 1$. We determine if the preceding three possibilities for the Jordan canonical form of M occur.

Let $\theta \in \mathbb{R}$ such that $e^{i\theta} \neq \pm 1$. If we choose $r_{\mathbf{x}}, r_{\mathbf{y}} \in \mathbb{R}$ such that $r_{\mathbf{x}} r_{\mathbf{y}} = \dfrac{2(1-\cos\theta)}{|\mathbf{x}^*S\mathbf{y}|^2} \neq 0$, then $\det(M) = 1 = \det(\operatorname{diag}(e^{i\theta}, e^{-i\theta}))$ and $\operatorname{tr}(M) = 2\cos\theta = \operatorname{tr}(\operatorname{diag}(e^{i\theta}, e^{-i\theta}))$. By Lemma 2, M is similar to $\operatorname{diag}(e^{i\theta}, e^{-i\theta})$.

If we choose $r_{\mathbf{x}}, r_{\mathbf{y}} \in \mathbb{R}$ such that $r_x\, r_y = \dfrac{4}{|\mathbf{x}^* S \mathbf{y}|^2}$, then $\mathrm{tr}(M) = -2 = \mathrm{tr}(J_2\,(-1))$ and $\det(M) = 1 = \det(J_2\,(-1))$. By Lemma 2, M is similar to $J_2(-1)$.

Let $\lambda \in \mathbb{R}$ such that $|\lambda| > 1$. If we choose $r_{\mathbf{x}}, r_{\mathbf{y}} \in \mathbb{R}$ such that $r_x\, r_y = \dfrac{(2-\lambda-\lambda^{-1})}{|\mathbf{x}^* S \mathbf{y}|^2} \neq 0$, then we have $\det(M) = 1 = \det(\mathrm{diag}(\lambda, \lambda^{-1}))$ and $\mathrm{tr}(M) = -2 = \mathrm{tr}(\mathrm{diag}(\lambda, \lambda^{-1}))$. Since $\lambda \neq \lambda^{-1}$, we have that M is similar to $\mathrm{diag}(\lambda, \lambda^{-1})$.

Theorem 3. Let $S \in GL_n$ *be indefinite Hermitian and* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *be given. If* $\{\mathbf{x}, \mathbf{y}\}$ *is linearly independent and* $H_{\mathbf{x}}, H_{\mathbf{y}} \in K_S$, *then the product* $H_{\mathbf{x}} H_{\mathbf{y}}$ is similar to one of the following:

a. $I_{n-4} \oplus J_2\,(1) \oplus J_2\,(1)$

b. $I_{n-2} \oplus J_2\,(-1)$

c. $I_{n-2} \oplus \mathrm{diag}(e^{i\theta}, e^{-i\theta})$, *where* $\theta \in \mathbb{R}$ *such that* $e^{i\theta} \neq \pm 1$

d. $I_{n-2} \oplus \mathrm{diag}(\lambda, \lambda^{-1})$, *where* $|\lambda| > 1$ *and* $\lambda \in \mathbb{R}$.

$H_{\mathbf{x}}, H_{\mathbf{y}} \in L_S$

We now consider the product of two elements of L_S. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent and $H_{\mathbf{x}}, H_{\mathbf{y}} \in L_S$, that is, $H_{\mathbf{x}} = I + \dfrac{e^{i\alpha}-1}{\mathbf{x}^* S\mathbf{x}}\mathbf{x}\mathbf{x}^* S$ and $H_y = I + \dfrac{e^{i\beta}-1}{\mathbf{y}^* S\mathbf{y}}\mathbf{y}\mathbf{y}^* S$, where $\mathbf{x}^* S\mathbf{x}$ and $\mathbf{y}^* S\mathbf{y}$ are nonzero, and $\alpha, \beta \in \mathbb{R}$ such that $e^{i\alpha} \neq 1$ and $e^{i\beta} \neq 1$. Since $H_{\mathbf{y}} = H_{a\mathbf{y}}$ for all nonzero $a \in \mathbb{C}$, we can assume that $\mathbf{x}^* S\mathbf{x}$, $\mathbf{y}^* S\mathbf{y} \in \{1, -1\}$. If $A = H_{\mathbf{x}}\, H_{\mathbf{y}}$, then

$$A = I + \frac{e^{i\alpha}-1}{\mathbf{x}^* S\mathbf{x}}\mathbf{x}\mathbf{x}^* S + \frac{e^{i\beta}-1}{\mathbf{y}^* S\mathbf{y}}\mathbf{y}\mathbf{y}^* S + \frac{e^{i\alpha}-1}{\mathbf{x}^* S\mathbf{x}}\frac{e^{i\beta}-1}{\mathbf{y}^* S\mathbf{y}}(\mathbf{x}^* S\mathbf{y})\mathbf{x}\mathbf{y}^* S.$$

Case 1: If $\mathbf{x}^* S\mathbf{y} = 0$, then $A = I + \dfrac{e^{i\alpha}-1}{\mathbf{x}^* S\mathbf{x}}\mathbf{x}\mathbf{x}^* S + \dfrac{e^{i\beta}-1}{\mathbf{y}^* S\mathbf{y}}\mathbf{y}\mathbf{y}^* S$. Observe that $A\mathbf{x} = \mathbf{x} + (e^{i\alpha} - 1)\mathbf{x} = e^{i\alpha}\mathbf{x}$. Hence, $\mathbf{x}$ is an eigenvector of A corresponding to $e^{i\alpha}$. Similarly, $\mathbf{y}$ is an eigenvector of A corresponding to $e^{i\beta}$. Since $\mathbf{x}^* S\mathbf{x}$ and $\mathbf{y}^* S\mathbf{y}$ are nonzero and $\mathbf{x}^* S\mathbf{y} = 0$, we have $\mathbb{C}^n = \mathrm{span}\{\mathbf{x}, \mathbf{y}\} \oplus \{\mathbf{x}, \mathbf{y}\}^S$. Hence A is similar to $I_{n-2} \oplus \mathrm{diag}(e^{i\alpha}, e^{i\beta})$.

Case 2: Suppose $\mathbf{x}^*S\mathbf{y} \neq 0$. We find any remaining eigenvalues of A. The images of $\mathbf{x}$ and $\mathbf{y}$ under A are

$$\begin{aligned} A\mathbf{x} &= \mathbf{x} + (e^{i\alpha} - 1)\mathbf{x} + \frac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{y}^*S\mathbf{x})\mathbf{y} + \frac{e^{i\alpha} - 1}{\mathbf{x}^*S\mathbf{x}}\frac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{x}^*S\mathbf{y})(\mathbf{y}^*S\mathbf{x})\mathbf{x} \\ &= \left(e^{i\alpha} + \frac{e^{i\alpha} - 1}{\mathbf{x}^*S\mathbf{x}}\frac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}}|\mathbf{x}^*S\mathbf{y}|^2\right)\mathbf{x} + \left(\frac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{y}^*S\mathbf{x})\right)\mathbf{y} \end{aligned}$$

and

$$\begin{aligned} A\mathbf{y} &= \mathbf{y} + \frac{e^{i\alpha} - 1}{\mathbf{x}^*S\mathbf{x}}(\mathbf{x}^*S\mathbf{y})\mathbf{x} + \frac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{y}^*S\mathbf{y})\mathbf{y} + \frac{e^{i\alpha} - 1}{\mathbf{x}^*S\mathbf{x}}\frac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{x}^*S\mathbf{y})(\mathbf{y}^*S\mathbf{y})\mathbf{x} \\ &= e^{i\beta}\mathbf{y} + \left(e^{i\beta}\mathbf{x}^*S\mathbf{y}\frac{e^{i\alpha} - 1}{\mathbf{x}^*S\mathbf{x}}\right)\mathbf{x}. \end{aligned}$$

Hence span$\{\mathbf{x}, \mathbf{y}\}$ is invariant under A. Consider the restriction of A to span$\{\mathbf{x}, \mathbf{y}\}$ and its matrix representation

$$L = \begin{bmatrix} e^{i\alpha} + \frac{e^{i\alpha} - 1}{\mathbf{x}^*S\mathbf{x}}\frac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}}|\mathbf{x}^*S\mathbf{y}|^2 & e^{i\beta}\mathbf{x}^*S\mathbf{y}\frac{e^{i\alpha} - 1}{\mathbf{x}^*S\mathbf{x}} \\ \frac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{y}^*S\mathbf{x}) & e^{i\beta} \end{bmatrix}$$

with respect to the ordered basis $\{\mathbf{x}, \mathbf{y}\}$.

Note that $a\mathbf{x} + b\mathbf{y} \in \{\mathbf{x}, \mathbf{y}\}^S$ for some $a, b \in \mathbb{C}$ if and only if $\mathbf{x}^*S(a\mathbf{x} + b\mathbf{y}) = 0$ and $\mathbf{y}^*S(a\mathbf{x} + b\mathbf{y}) = 0$, that is $\begin{bmatrix} \mathbf{x}^*S\mathbf{x} & \mathbf{x}^*S\mathbf{y} \\ \mathbf{y}^*S\mathbf{x} & \mathbf{y}^*S\mathbf{y} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $\mathbf{x}^*S\mathbf{x}, \mathbf{y}^*S\mathbf{y} \in \{1, -1\}$, we have $\{\mathbf{x}, \mathbf{y}\}^S \cap \text{span}\{\mathbf{x}, \mathbf{y}\} = \{\mathbf{0}\}$ if and only if $\mathbf{x}^*S\mathbf{x}$ and $\mathbf{y}^*S\mathbf{y}$ have opposite signs or $|\mathbf{x}^*S\mathbf{y}| \neq 1$.

If $\mathbf{x}^*S\mathbf{x} = \mathbf{y}^*S\mathbf{y} \in \{\pm 1\}$ and $|\mathbf{x}^*S\mathbf{y}| = 1$, then $\mathbf{x}, \mathbf{y} \notin \{\mathbf{x}, \mathbf{y}\}^S$ and

$$\{\mathbf{x}, \mathbf{y}\}^S \cap \text{span}\{\mathbf{x}, \mathbf{y}\} = \text{span}\{(\mathbf{x}^*S\mathbf{y})\mathbf{x} - (\mathbf{x}^*S\mathbf{x})\mathbf{y}\}.$$

Hence span$\{\mathbf{x}\} \oplus \{\mathbf{x}, \mathbf{y}\}^S$ is of dimension $n-1$ and contains span$\{\mathbf{x}, \mathbf{y}\}$. Now $A\mathbf{x}$ can be rewritten as $A\mathbf{x} = e^{i(\alpha+\beta)}\,\mathbf{x} - (e^{i\beta} - 1)(\mathbf{y}^*S\mathbf{x})[(\mathbf{x}^*S\mathbf{y})\mathbf{x} - (\mathbf{x}^*S\mathbf{x})\mathbf{y}]$. This implies that span$\{\mathbf{x}\} \oplus \{\mathbf{x}, \mathbf{y}\}^S$ is invariant under A. Since $\det A = e^{i(\alpha+\beta)}$ and rank$(A - I) = 2$, we have that A is similar to $I_{n-3} \oplus J_2(1) \oplus [e^{i(\alpha+\beta)}]$, if $e^{i(\alpha+\beta)} \neq 1$; or $I_{n-3} \oplus J_3(1)$, if $e^{i(\alpha+\beta)} = 1$.

If $\mathbb{C}^n = \text{span}\{\mathbf{x}, \mathbf{y}\} \oplus \{\mathbf{x}, \mathbf{y}\}^S$, then A is similar to $I_{n-2} \oplus L$ and 1 is not an eigenvalue of L. Observe that $\det L = e^{i(\alpha+\beta)}$ and $\operatorname{tr} L = e^{i\alpha} + e^{i\beta} + \dfrac{e^{i\alpha} - 1}{\mathbf{x}^*S\mathbf{x}} \dfrac{e^{i\beta} - 1}{\mathbf{y}^*S\mathbf{y}} |\mathbf{x}^*S\mathbf{y}|^2$.

Since $A \in U_S$ and L is not a scalar matrix, then L is similar to one of the following:

diag$(e^{i\theta}, e^{i\phi})$, where $\theta, \phi \in \mathbb{R}$ such that $e^{i\theta}, e^{i\phi}$ are distinct and both are not equal to 1; $J_2(\lambda)$, where $|\lambda| = 1$ but $\lambda \neq 1$; or diag $(\lambda, \overline{\lambda^{-1}})$, where $|\lambda| > 1$.

It suffices to determine whether the last two possibilities for the Jordan canonical form of L occur. But first we need to determine the possible nonzero values of $\mathbf{x}^*S\mathbf{y}$, when $\mathbf{x}^*S\mathbf{x}$, $\mathbf{y}^*S\mathbf{y} \in \{1, -1\}$ and $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent. Let $\mathbf{e}_i \in \mathbb{C}^n$ denote the column with ith entry 1 and 0 elsewhere. Suppose $c \in \mathbb{C}$ is nonzero and $S = P^*(I_k \oplus -I_{n-k})P$, for some nonsingular P and integer $0 < k < n$. If $|c| > 1$, we can take $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $P\mathbf{x} = \mathbf{e}_1$ and $P\mathbf{y} = c\mathbf{e}_1 + \sqrt{|c|^2 - 1}\,\mathbf{e}_{k+1}$, so that $\mathbf{x}^*S\mathbf{x} = 1$, $\mathbf{y}^*S\mathbf{y} = |c|^2 - (|c|^2 - 1) = 1$, and $\mathbf{x}^*S\mathbf{y} = c$. Thus, if $|c| > 1$, there exists a linearly independent set $\{\mathbf{x}, \mathbf{y}\}$ such that $\mathbf{x}^*S\mathbf{x} = \mathbf{y}^*S\mathbf{y}$ and $\mathbf{x}^*S\mathbf{y} = c$. If $c \in \mathbb{C}$ is nonzero and we take $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $P\mathbf{x} = \mathbf{e}_1$ and $P\mathbf{y} = c\mathbf{e}_1 + \sqrt{|c|^2 - 1}\,\mathbf{e}_{k+1}$, then $\mathbf{x}^*S\mathbf{x} = 1$, $\mathbf{y}^*S\mathbf{y} = |c|^2 - (|c|^2 + 1) = -1$ and $\mathbf{x}^*S\mathbf{y} = c$. Hence every nonzero $c \in \mathbb{C}$ can be realized as $\mathbf{x}^*S\mathbf{y}$ by a linearly independent set $\{\mathbf{x}, \mathbf{y}\}$ such that $\mathbf{x}^*S\mathbf{x} = -\mathbf{y}^*S\mathbf{y}$, when S is *-congruent to $I_k \oplus -I_{n-k}$.

Let $\alpha = \beta \in \mathbb{R}$ such that $\alpha \neq k\pi$, for all $k \in \mathbb{Z}$. If $a = \operatorname{Re}(e^{i\alpha})$, then $\dfrac{-4e^{i\alpha}}{(e^{i\alpha} - 1)^2} = \dfrac{2}{1-a} > 1$. If we take $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $\mathbf{x}^*S\mathbf{x} = 1 = \mathbf{y}^*S\mathbf{y}$ and $|\mathbf{x}^*S\mathbf{y}|^2 = \dfrac{-4e^{i\alpha}}{(e^{i\alpha} - 1)^2}$, then $\operatorname{tr} L = 2e^{i\alpha} + (e^{i\alpha} - 1)^2 |\mathbf{x}^*S\mathbf{y}|^2 = -2e^{i\alpha}$ and $\det L = e^{i2\alpha}$. Since L is not a scalar matrix, it follows from Lemma 2 that L is similar to $J_2(-e^{i\alpha})$, where $e^{i\alpha} \neq \pm 1$.

If we take $e^{i\alpha} = e^{-i\beta} = i$, and $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $\mathbf{x}^*S\mathbf{x} = 1 = -\mathbf{y}^*S\mathbf{y}$ and $|\mathbf{x}^*S\mathbf{y}| = 1$, then $\operatorname{tr} L = -2$ and $\det L = 1$. Since L is not a scalar matrix, L is similar to $J_2(-1)$.

Let $\lambda = re^{i\theta}$, where $r > 1$ and $\theta \neq 2k\pi$ for all $k \in \mathbb{Z}$. Then $-\dfrac{e^{i\theta}(r-1)^2}{(e^{i\theta} - 1)^2 r}$ is positive.

If we take $\alpha = \beta = \theta$, and $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $\mathbf{x}^*S\mathbf{x} = 1 = -\mathbf{y}^*S\mathbf{y}$ and $|\mathbf{x}^*S\mathbf{y}|^2 = -\dfrac{e^{i\theta}(r-1)^2}{(e^{i\theta}-1)^2 r}$, then tr $L = 2e^{i\theta} - (e^{i\theta}-1)^2\ |\mathbf{x}^*S\mathbf{y}|^2 = e^{i\theta}\ (r + r^{-1}) = \lambda + \overline{\lambda^{-1}}$ and det $L = e^{i2\theta} = \lambda\,\overline{\lambda^{-1}}$. Hence L is similar to diag$(\lambda, \overline{\lambda^{-1}})$.

Let $\lambda = r$, where $r > 1$. Let $\beta = -\alpha$ and $\alpha \in \mathbb{R}$ such that $\mathrm{Re}(e^{i\alpha}) = r^{-1}$. Since $\dfrac{(r-1)^2}{r} > 0$, we have $\dfrac{r - r^{-1}}{2(1-r^{-1})} > 1$. If we take $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $\mathbf{x}^*S\mathbf{x} = 1 = \mathbf{y}^*S\mathbf{y}$ and $|\mathbf{x}^*S\mathbf{y}|^2 = \dfrac{r - r^{-1}}{2(1-r^{-1})}$, then tr $L = 2r^{-1} + 2(1-r^{-1})\ |\mathbf{x}^*S\mathbf{y}|^2 = r + r^{-1}$ and det $L = 1$. Hence L is similar to diag(r, r^{-1}).

Theorem 4. Let $S \in GL_n$ *be indefinite Hermitian and* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *be given. If* $\{\mathbf{x}, \mathbf{y}\}$ *is linearly independent such that* $H_{\mathbf{x}}, H_{\mathbf{y}} \in L_S$, *then the product* $H_{\mathbf{x}}H_{\mathbf{y}}$ *is similar to one of the following:*

a. $I_{n-2} \oplus \mathrm{diag}(e^{i\theta}, e^{i\phi})$, *where* $\theta, \phi \in \mathbb{R}$ *such that* $e^{i\theta}, e^{i\phi} \neq 1$

b. $I_{n-3} \oplus J_2(1) \oplus [e^{i\theta}]$, *where* $\theta \in \mathbb{R}$ *and* $e^{i\theta} \neq 1$

c. $I_{n-3} \oplus J_3(1)$

d. $I_{n-2} \oplus J_2(\lambda)$, *where* $|\lambda| = 1$ *and* $\lambda \neq 1$

e. $I_{n-2} \oplus \mathrm{diag}(\lambda, \overline{\lambda^{-1}})$, *where* $|\lambda| > 1$.

$H_x \in K_S$ and $H_y \in L_S$

Lastly, we consider the product of an element of K_S and of L_S. If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ are nonzero such that $H_{\mathbf{x}} \in K_S$ and $H_{\mathbf{y}} \in L_S$, then $H_{\mathbf{x}} = I + ir\mathbf{x}\mathbf{x}^*S$, where $r \in \mathbb{R} \setminus \{0\}$, and $\mathbf{x}^*S\mathbf{x} = 0$, and $Hy = I + \dfrac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}\ \mathbf{y}\mathbf{y}^*S$, where $e^{i\alpha} \neq 1$. Note that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent since $\mathbf{x}^*S\mathbf{x} = 0 \neq \mathbf{y}^*S\mathbf{y}$. If $A = H_x H_y$, then

$$A = I + ir\mathbf{x}\mathbf{x}^*S + \frac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}\ \mathbf{y}\mathbf{y}^*S + ir\,\frac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}\ (\mathbf{x}^*S\mathbf{y})\ \mathbf{x}\mathbf{y}^*S.$$

Case 1: If $\mathbf{x}^*S\mathbf{y} = 0$, then $A = I + ir\mathbf{x}\mathbf{x}^*S + \dfrac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}\ \mathbf{y}\mathbf{y}^*S$, $A\mathbf{x} = \mathbf{x}$, and $A\mathbf{y} = e^{i\alpha}\mathbf{y}$. Since $\mathbf{x}^*S\mathbf{y} = 0$ and $\mathbf{x}^*S\mathbf{x} = 0$, we have $(A - I)^2 = \dfrac{(e^{i\alpha}-1)^2}{\mathbf{y}^*S\mathbf{y}}\ \mathbf{y}\mathbf{y}^*S$ and $(A - I)^3 = \dfrac{(e^{i\alpha}-1)^3}{\mathbf{y}^*S\mathbf{y}}\ \mathbf{y}\mathbf{y}^*S$.

Observe that rank$(A - I) = 2$ and rank $(A - I)^2 =$ rank$(A - I)^3 = 1$, which imply that 2 is the size of the largest Jordan block corresponding to 1, and the number of Jordan blocks of size 2 corresponding to 1 is rank$(A - I)$ − 2rank$(A - I)^2$ + rank$(A - I)^3$ = 2 − 2(1) + 1 = 1. Since there are $n-2$ Jordan blocks corresponding to 1 and $\det A = e^{i\alpha}$, we have that A is similar to $I_{n-3} \oplus J_2(1) \oplus [e^{i\alpha}]$.

Case 2: Suppose $\mathbf{x}^*S\mathbf{y} \neq 0$. The images of $\mathbf{x}$ and $\mathbf{y}$ under A are

$$\begin{aligned} A\mathbf{x} &= \mathbf{x} + \frac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{y}^*S\mathbf{x})\mathbf{y} + ir\frac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}|\mathbf{x}^*S\mathbf{y}|^2\mathbf{x} \\ &= \left(1+ir\frac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}|\mathbf{x}^*S\mathbf{y}|^2\right)\mathbf{x} + \frac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{y}^*S\mathbf{x})\mathbf{y} \end{aligned}$$

and

$$A\mathbf{y} = \mathbf{y} + ir(\mathbf{x}^*S\mathbf{y})\mathbf{x} + (e^{i\alpha} - 1)\mathbf{y} + ir(e^{i\alpha} - 1)(\mathbf{x}^*S\mathbf{y})\mathbf{x} = ire^{i\alpha}(\mathbf{x}^*S\mathbf{y})\mathbf{x} + e^{i\alpha}\mathbf{y}.$$

Hence span$\{\mathbf{x}, \mathbf{y}\}$ is invariant under A. Consider the restriction of A to span$\{\mathbf{x}, \mathbf{y}\}$ and its matrix representation

$$K = \begin{bmatrix} 1 + ir\dfrac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}|\mathbf{x}^*S\mathbf{y}|^2 & ire^{i\alpha}(\mathbf{x}^*S\mathbf{y}) \\ \dfrac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}(\mathbf{y}^*S\mathbf{x}) & e^{i\alpha} \end{bmatrix}$$

with respect to the ordered basis $\{\mathbf{x}, \mathbf{y}\}$. Since $\mathbb{C}^n = \text{span}\{\mathbf{x}, \mathbf{y}\} \oplus \{\mathbf{x}, \mathbf{y}\}^S$, A is similar to $I_{n-2} \oplus K$ and 1 is not an eigenvalue of K. Note that $\det K = e^{i\alpha} \neq 1$ and $\operatorname{tr} K = e^{i\alpha} + 1 + ir\dfrac{e^{i\alpha}-1}{\mathbf{y}^*S\mathbf{y}}|\mathbf{x}^*S\mathbf{y}|^2$. Since A is S-unitary, K is similar to one of the following: diag$(e^{i\theta}, e^{i\phi})$, where $\theta, \phi \in \mathbb{R}$ such that $e^{i\theta}, e^{i\phi}$ are distinct with both not equal to 1, and $e^{i(\theta+\phi)} = e^{i\alpha}$; or diag$(\lambda, \overline{\lambda^{-1}})$, where $|\lambda| > 1$ and $\lambda \neq \pm 1$.

We now determine whether the three possibilities for the Jordan canonical form of K occur.

Let θ, $\phi \in \mathbb{R}$ such that $e^{i\theta}$, $e^{i\phi}$, and $e^{i(\theta+\phi)}$ are not equal to 1, and $e^{i\theta} \neq e^{i\phi}$. If $\alpha = \theta + \phi$, choose r $\in \mathbb{R}$ such that $r(e^{i\alpha} - 1) = \dfrac{(\mathbf{y}^* S\mathbf{y})(1-e^{i\theta})(e^{i\phi}-1)}{i|\mathbf{x}^* S\mathbf{y}|^2}$. This has a solution since $\dfrac{(1-e^{i\theta})(e^{i\phi}-1)}{e^{i\alpha}-1} = -1 + \dfrac{e^{i\theta}+e^{i\phi}-2}{e^{i\alpha}-1}$ is nonzero and the real part of $\dfrac{e^{i\theta}+e^{i\phi}-2}{e^{i\alpha}-1}$ is 1. Then det $K = e^{i(\theta+\phi)} = \det(\mathrm{diag}(e^{i\theta}, e^{i\phi}))$ and tr $K = e^{i\theta} + e^{i\phi} = \mathrm{tr}(\mathrm{diag}(e^{i\theta}, e^{i\phi}))$. Thus K is similar to $\mathrm{diag}(e^{i\theta}, e^{i\phi})$.

Let $\lambda = te^{i\gamma}$, where $t, \gamma \in \mathbb{R}$ such that $t > 1$ and $e^{i2\gamma} \neq 1$. Choose $\alpha = 2\gamma$ and $r \in \mathbb{R}$ such that $r(e^{i\alpha} - 1) = \dfrac{(\mathbf{y}^* S\mathbf{y})\,(1-te^{i\gamma})\,(t^{-1}e^{i\gamma}-1)}{i|\mathbf{x}^* S\mathbf{y}|^2}$. This has a solution since $\dfrac{(1-te^{i\gamma})\,(t^{-1}e^{i\gamma}-1)}{e^{i\alpha}-1} = -1 + \dfrac{(t+t^{-1})\,e^{i\gamma}-2}{e^{i2\gamma}-1}$ is nonzero and the real part of $\dfrac{(t+t^{-1})\,e^{i\gamma}-2}{e^{i2\gamma}-1}$ is 1. Then det $K = \lambda\,\overline{\lambda^{-1}} = \det(\mathrm{diag}(\lambda, \overline{\lambda^{-1}}))$ and tr $K = (t + t^{-1})\,e^{i\gamma} = \mathrm{tr}\,(\mathrm{diag}(\lambda, \overline{\lambda^{-1}}))$. By Lemma 2, K is similar to $\mathrm{diag}(\lambda, \overline{\lambda^{-1}})$.

Let $\lambda = e^{i\beta}$, where $\beta \in \mathbb{R}$ and $\lambda \neq \pm 1$. Choose $\alpha = 2\beta$ and $r \in \mathbb{R}$ such that $r = \dfrac{(1-\lambda)\,\mathbf{y}^* S\mathbf{y}}{i(\lambda+1)|\mathbf{x}^* S\mathbf{y}|^2}$. This has a solution since $\dfrac{1-\lambda}{\lambda+1} = -1 + \dfrac{2}{\lambda+1}$ and the real part of $\dfrac{2}{\lambda+1}$ is 1. Then det $K = \lambda^2 = \det J_2(\lambda)$ and tr $K = 2\lambda = \mathrm{tr}\, J_2(\lambda)$. Since K is not a scalar matrix, K is similar to $J_2(\lambda)$.

Theorem 5. Let $S \in GL_n$ *be indefinite Hermitian and* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *be given. If* $\{\mathbf{x}, \mathbf{y}\}$ *is linearly independent such that* $H_{\mathbf{x}} \in K_S$ *and* $H_{\mathbf{y}} \in L_S$, *then the product* $H_{\mathbf{x}} H_{\mathbf{y}}$ *is similar to one of the following:*

a. $I_{n-3} \oplus J_2(1) \oplus [e^{i\alpha}]$, *for some* $\alpha \in \mathbb{R}$ *such that* $e^{i\alpha} \neq 1$

b. $I_{n-2} \oplus \mathrm{diag}(e^{i\theta}, e^{i\phi})$, *where* $\theta, \phi \in \mathbb{R}$ *such that* $ei^{\theta}, e^{i\phi}, e^{i(\theta+\phi)}$ *are all not equal to* 1, *and* $e^{i\theta} \neq e^{i\phi}$

c. $I_{n-2} \oplus \mathrm{diag}(\lambda, \overline{\lambda^{-1}})$, *where* $|\lambda| > 1$ *and* $\lambda \notin \mathbb{R}$

d. $I_{n-2} \oplus J_2(\lambda)$, *where* $|\lambda| = 1$ *but* $\lambda \neq \pm 1$.

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